Nonleptonic B decays into two light mesons in soft-collinear effective theory

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Abstract

We consider nonleptonic B decays into two light mesons at leading order in soft-collinear effective theory, and show that the decay amplitudes are factorized to all orders in α_s. The operators for nonleptonic B decays in the full theory are first matched to the operators in $\mathrm{SCET_I}$, which is the effective theory appropriate for $\sqrt{m_b\Lambda} < \mu < m_b$ with $\Lambda \sim 0.5$ GeV. We evolve the operators and the relevant time-ordered products in $\mathrm{SCET_I}$ to $\mathrm{SCET_{II}}$, which is appropriate for $\mu < \sqrt{m_b\Lambda}$. Using the gauge-invariant operators in $\mathrm{SCET_{II}}$, we compute nonleptonic B decays in SCET, including the nonfactorizable spectator contributions and spectator contributions to the heavy-to-light form factor. As an application, we present the decay amplitudes for $\overline{B} \to \pi\pi$ in soft-collinear effective theory.

1 Introduction

Decay of B mesons plays an important role in particle physics since it is a testing ground for the Standard Model and a window for possible new physics. We can obtain good information on the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements and CP violation from B decays. Since the contribution of QCD in B decays changes the whole structure of the theory, the study of B decays is an intertwined field of particle physics. Among these decays, nonleptonic B decays have been the subject of intense interest. Especially the treatment of nonperturbative effects from the strong interaction is a serious theoretical problem in nonleptonic decays. Precise experimental observation of nonleptonic B decays makes it urgent to give firm theoretical prediction including the effects of CP violation. There has been a lot of theoretical progress in nonleptonic B decays and we suggest how to consider nonleptonic B decays from the viewpoint of the soft-collinear effective theory (SCET).

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The effective Hamiltonian for nonleptonic decays from the Standard Model has been derived and the Wilson coefficients of the operators for B decays have been calculated to next-to-leading order, and next-to-next-to-leading order for some operators [1]. In order to evaluate the hadronic matrix elements of four-quark operators, naive factorization [2] was assumed, in which the matrix elements were reduced to products of current matrix elements. But there was no justification for this assumption except the argument of color transparency [3]. Besides that, decay amplitudes depend on an arbitrary renormalization scale μ in naive factorization since the Wilson coefficients depend on μ, while the matrix elements of operators do not. Ali et al. [4] have improved the problem of the scale dependence by including radiative corrections of the operators before taking hadronic matrix elements. Then the μ dependence of the Wilson coefficients is cancelled by that of the radiative corrections of the operators. However, the decay amplitudes depend on calculation schemes since the off-shell renormalization scheme is used [5].

Politzer and Wise [6] suggested to take the heavy quark mass limit in computing corrections to the decay rate ratio $\Gamma(\overline{B} \to D^*\pi)/\Gamma(\overline{B} \to D\pi)$. They have considered radiative corrections for nonfactorizable contributions and found that they are finite and the decay amplitudes are factorized. Beneke et al. [7] have extended this idea to general two-body decays including two light final-state mesons, in which the decay amplitudes can be expanded in a power series of $1/m_b$. They show that nonfactorizable contributions including spectator interactions are factorized as a convolution of the hard scattering amplitudes with the meson wave functions, and the corrections are suppressed by powers of $1/m_b$. This is an important step toward theoretical understanding of nonleptonic B decays. First the amplitudes can be obtained from first principles and a systematic $1/m_b$ expansion is possible. Second, since the on-shell renormalization is used, there is no scheme dependence. And they improved previous approaches by including momentum-dependent parts, which had not been included previously. However, when higher-twist light-cone wave functions are included, there appear infrared divergences in the amplitudes. These are treated as theoretical uncertainties, but from the theoretical viewpoint it is a problem to be solved in this approach.

Bauer et al. [8,9] have proposed an effective field theory in which massless quarks move with large energy. This effective theory, called the soft-collinear effective theory, is appropriate for light quarks with large energy. It has been applied to hard scattering processes and B decays [10–15]. It is also an appropriate effective theory for nonleptonic B decays to two light mesons. In this paper, we apply SCET to nonleptonic B decays into light mesons. We construct all the relevant operators in the effective theory at leading order in a gauge-invariant way by integrating out all the off-shell modes. The Wilson coefficients are calculated by matching the full theory onto SCET. We show that the four-quark operators in SCET are factorized to all orders in α_s

and the argument of color transparency is explictly shown. And we consider nonfactorizable spectator contribution in SCET, and find that they are also factorized to all orders in α_s. The basic idea of the factorization properties in B decays into two light mesons was sketched in Ref. [16]. In this paper, we extend the argument and discuss intricate characteristics of SCET in nonleptonic decays, the details of the procedure of matching, and present all the Wilson coefficients and technical details in the calculation. We also present the analysis of $B \to \pi\pi$ decays as an application, which is consistent with the calculation in the heavy quark mass limit at lowest order in α_s.

The organization of the paper is as follows: In Section 2, we consider the effective theories $\mathrm{SCET_I}$ and $\mathrm{SCET_{II}}$, which we employ in the analysis of nonleptonic B decays and explain the field contents and discuss how to match these effective theories. In Section 3, we construct four-quark operators in $\mathrm{SCET_I}$, which are gauge invariant by integrating out off-shell modes. This is achieved by attaching gluons to fermion legs and by integrating out off-shell intermediate states. We discuss the factorization properties of the operators. In Section 4, we match the full theory and $\mathrm{SCET_I}$ and obtain the Wilson coefficients of the four-quark operators. In Section 5, we consider nonfactorizable spectator interaction, in which subleading operators are enhanced to give the leading-order result. The structure of the subleading operators and the mechanism for the enhancement are determined by the power counting and the matching between effective theories. In Section 6, we consider the contributions to the heavy-to-light form factor. The time-ordered products of subleading operators from the heavy-to-light current and the interaction of an ultrasoft (usoft) quark and a collinear quark contribute to the form factor. In Section 7 we combine all the results to apply SCET to nonleptonic decays $B \to \pi\pi$. And finally a conclusion is presented. In Appendix A, we show how the auxiliary field method can be used to derive gauge-invariant four-quark operators in $\mathrm{SCET_{II}}$. In Appendix B, we employ the auxiliary field method to derive gauge-invariant subleading operators in $\mathrm{SCET_I}$.

2 Construction of the effective theories $\mathrm{SCET_I}$ and $\mathrm{SCET_{II}}$

The formalism of SCET and the procedure of the two-step matching are discussed comprehensively in Refs. [9,15] and in Ref. [17]. We will not discuss them in detail, but we review them briefly here. The momentum of an energetic quark moving in the light-cone direction n^μ can be decomposed as

$$p^\mu = \frac{n^\mu}{2}\overline{n}\cdot p + p_\perp^\mu + \frac{\overline{n}^\mu}{2} n\cdot p = \mathcal{O}(\lambda^0) + \mathcal{O}(\lambda) + \mathcal{O}(\lambda^2), \tag{1}$$

where n^μ, $\overline{n}^\mu$ are two light-like vectors satisfying $n^2 = \overline{n}^2 = 0$, $n \cdot \overline{n} = 2$. There are three scales $\overline{n} \cdot p \sim Q$, $p_\perp^\mu \sim Q\lambda$, and $n \cdot p \sim Q\lambda^2$, where Q is the hard scale. The momentum squared p^2 is typically of order $Q\Lambda$, where Λ is a typical hadronic scale $\Lambda \sim 0.5$ GeV. We introduce a small expansion parameter $\lambda \sim \sqrt{\Lambda/Q} \ll 1$ to facilitate the power counting. We construct SCET in two steps, namely $\mathrm{SCET_I}$ for $\sqrt{Q\Lambda} < \mu < Q$, and $\mathrm{SCET_{II}}$ for $\mu < \sqrt{Q\Lambda}$. Formally we can construct $\mathrm{SCET_{II}}$ directly from the full theory, but it is conceptually easy to construct $\mathrm{SCET_I}$ and $\mathrm{SCET_{II}}$ successively.

In SCET, the collinear quarks ξ and gluons A_n^μ have momenta $p^\mu = (n \cdot p, \overline{n} \cdot p, p_\perp) \sim Q(\lambda^2, 1, \lambda)$. There are the soft fields q_s, and A_s^μ, with momenta $p_s^\mu \sim Q\lambda$, and the usoft fields q_{us}, A_{us}^μ with momenta $p_{us}^\mu \sim Q\lambda^2$. The collinear quarks interact with collinear gluons or usoft gluons, but not with soft gluons since they make collinear particles or heavy quarks off the mass shell. In order to derive the power counting method in $\mathrm{SCET_I}$, we move all the dependence on λ to the interaction vertices, and determine the λ dependence of all the fields [9]. Then we can construct all the operators in powers of λ systematically by integrating out the off-shell modes of order Q. The guiding principles of constructing operators in SCET are gauge invariance [15], reparameterization invariance [13,14] and the power counting [18]. The Wilson coefficients of the operators can be obtained through the matching between the full theory and $\mathrm{SCET_I}$ by requiring that the matrix elements of operators in both theories be the same at any order of the perturbation theory. There is no mixing of operators with different powers of λ through radiative corrections as long as $Q \gg \Lambda$ since the matching is performed perturbatively.

In $\mathrm{SCET_{II}}$ for $\mu < \sqrt{Q\Lambda}$, we integrate out all the off-shell modes of order $\sqrt{Q\Lambda}$. Then the collinear fields in $\mathrm{SCET_{II}}$ have momenta $p^\mu \sim Q(\lambda'^2, 1, \lambda')$, and the soft degrees of freedom have momenta $p_s \sim Q\lambda'$, where $\lambda' \sim \Lambda/Q$ is a new small expansion parameter in $\mathrm{SCET_{II}}$. Here the collinear modes have momenta $p^2 \sim \Lambda^2$, which is appropriate to describe hadrons of size $\sim 1/\Lambda$. The degrees of freedom of order $\sim \sqrt{Q\Lambda}$ in $\mathrm{SCET_I}$ are all integrated out, and the usoft degrees of freedom in $\mathrm{SCET_I}$ of order $p_{us} \sim Q\lambda^2$ remain in $\mathrm{SCET_{II}}$ with momentum $p_s \sim Q\lambda^2 = Q\lambda'$, and we call them soft degrees of freedom in $\mathrm{SCET_{II}}$. The matching of $\mathrm{SCET_I}$ onto $\mathrm{SCET_{II}}$ is performed at $\mu \sim \sqrt{Q\Lambda}$.

A subtle technical point in matching is to require gauge invariance. In order to see how to keep the gauge invariance, let us consider a simple example of a heavy-to-light current. In the full theory the current $\overline{q}\Gamma b$ is matched to the operator in $\mathrm{SCET_I}$ as

$$C(n \cdot \mathcal{P})\overline{\xi}\Gamma W h, \tag{2}$$

where $C(n \cdot \mathcal{P})$ is the Wilson coefficient, and $\mathcal{P}^\mu = \frac{1}{2} n^\mu n \cdot \mathcal{P} + \mathcal{P}_\perp^\mu$ is the label momentum operator. The factor W is the Wilson line which is given by

$$W = \sum_{\text{perm}} \exp\Big[-g \frac{1}{\overline{n} \cdot \mathcal{P}} \overline{n} \cdot A_n\Big]. \tag{3}$$

This Wilson line is obtained by attaching collinear gluons to the heavy quark and integrate out the off-shell modes of the intermediate states of order Q.

In SCET_{II}, the emission of soft gluons $p^\mu \sim Q\lambda'$ makes the collinear fields off the mass shell and we have to integrate out the off-shell modes. This is achieved by attaching usoft gluons to external fermion lines in SCET_{I} and by integrating out the off-shell modes of order $Q\lambda^2 = Q\lambda'$. It corresponds to factorizing the usoft-collinear interactions with the field redefinition [15]

$$\xi^{(0)} = Y^\dagger \xi, \quad A_n^{(0)} = Y^\dagger A_n Y, \quad Y(x) = \mathrm{P} \exp\Big(ig \int_{-\infty}^{x} ds n \cdot A_{us}(ns)\Big). \tag{4}$$

The collinear effective Lagrangian can be written in terms of $\xi^{(0)}$ and $A_n^{(0)\mu}$, that is, the collinear-usoft interactions are factorized with the field redefinitions given in Eq. (4). Then we match SCET_{I} onto SCET_{II} at a scale $\mu \sim \sqrt{Q\Lambda}$. For example, the heavy-to-light current is matched as

$$\overline{q}\Gamma b \to C(\overline{n} \cdot \mathcal{P}) \overline{\xi} W \Gamma h \to C(\overline{n} \cdot \mathcal{P}) \overline{\xi}^{(0)} W^{(0)} \Gamma Y^\dagger h, \tag{5}$$

where $W^{(0)} = Y^\dagger W Y$. In SCET_{II}, the fields with the superscript (0) are fundamental objects. We rename the usoft field as the soft field in SCET_{II}. We write $Y^\dagger h \to S^\dagger h$, and lower the off-shellness of the collinear fields.

Since the leading collinear Lagrangian is the same in SCET_{I} and SCET_{II}, we can simply replace $C(\overline{n} \cdot \mathcal{P}) \overline{\xi}^{(0)} W^{(0)} \to C(\overline{n} \cdot \mathcal{P}) \overline{\xi}^{\text{II}} W^{\text{II}}$, and we obtain the final form of the operator

$$\overline{q}\Gamma b \to C(\overline{n} \cdot \mathcal{P}) \overline{\xi}^{\text{II}} W^{\text{II}} \Gamma S^\dagger h, \tag{6}$$

where the superscript II, which denotes SCET_{II}, will be dropped from now on. From the two-step matching, it is clear why the Wilson coefficients do not depend on the soft momentum.

The advantage of the two-step matching is manifest when we consider time-ordered products in SCET_{I} since these can induce jet functions involving the fluctuations of order $p^2 \sim Q\Lambda$. In SCET_{I} we can compute the jet functions with a well-defined set of Feynman rules independent of the computation of the Wilson coefficients at $p^2 = Q^2$. And the scaling of operators in $\mathrm{SCET_{II}}$ is constrained by the power counting in $\mathrm{SCET_I}$ and especially, $\mathrm{SCET_I}$ puts a constraint of the number of factors of $1/\Lambda$, which can be induced from the fluctuations $1/Q\Lambda$. Therefore we can know how many powers of $1/Q\Lambda$ appear for a given process at a given order in α_s, and this power counting is not spoiled by the loop effects since there is no dependence on the soft momentum in the Wilson coefficients. This feature will be seen explicitly when we evaluate the decay amplitudes of nonleptonic decays which involve time-ordered products.

In nonleptonic decays into two light mesons, there are collinear quarks moving in opposite directions. We choose these two directions as n^μ and $\overline{n}^\mu$. For an energetic quark moving in the $\overline{n}^\mu$ direction, the momentum is decomposed as

$$p_{\overline{n}}^\mu = \frac{n \cdot p_{\overline{n}}}{2}\overline{n}^\mu + p_{\overline{n}\perp}^\mu + \frac{\overline{n} \cdot p_{\overline{n}}}{2} n^\mu = \mathcal{O}(\lambda^0) + \mathcal{O}(\lambda) + \mathcal{O}(\lambda^2). \tag{7}$$

We denote a collinear quark by ξ (χ) which moves in the n^μ ($\overline{n}^\mu$) direction. These fields satisfy the relations

$$\not{n}\xi = 0, \quad \frac{\not{n}\not{\overline{n}}}{4}\xi = \xi, \quad \not{\overline{n}}\chi = 0, \quad \frac{\not{\overline{n}}\not{n}}{4}\chi = \chi. \tag{8}$$

The collinear gauge field in the n^μ ($\overline{n}^\mu$) direction is written as A_n^μ ($A_{\overline{n}}^\mu$). The effective Lagrangian for χ and $A_{\overline{n}}^\mu$, which can be obtained from the colllinear Lagrangian for ξ and A_n^μ by replacing $\xi \leftrightarrow \chi$, $n^\mu \leftrightarrow \overline{n}^\mu$ respectively. This situation is simlar to two jets in the opposite direction [12], but we have a heavy quark interacting with collinear gluons in both directions, which makes the analysis more interesting and complicated.

3 Construction of gauge-invariant four-quark operators in SCET

The effective Hamiltonian for B decays in the full theory is given as

$$H_{\mathrm{eff}} = \frac{G_F}{\sqrt{2}} \sum_{p=u,c} V_{pd}^* V_{pb} \Big(C_1 O_1^p + C_2 O_2^p + \sum_{i=3,\cdots,6,8} C_i O_i \Big), \tag{9}$$

where the local $\Delta B = 1$ operators are given by

$$\begin{aligned} O_1^p &= (\overline{p}_\alpha b_\alpha)_{V-A} (\overline{d}_\beta p_\beta)_{V-A}, \quad O_2^p = (\overline{p}_\beta b_\alpha)_{V-A} (\overline{d}_\alpha p_\beta)_{V-A}, \\ O_3 &= (\overline{d}_\alpha b_\alpha)_{V-A} \sum_q (\overline{q}_\beta q_\beta)_{V-A}, \quad O_4 = (\overline{d}_\beta b_\alpha)_{V-A} \sum_q (\overline{q}_\alpha q_\beta)_{V-A}, \end{aligned}$$

$$O_5 = (\overline{d}_\alpha b_\alpha)_{V-A} \sum_q (\overline{q}_\beta q_\beta)_{V+A}, \quad O_6 = (\overline{d}_\beta b_\alpha)_{V-A} \sum_q (\overline{q}_\alpha q_\beta)_{V+A},$$

$$O_8 = \frac{-g}{8\pi^2} m_b \overline{d}_\alpha \sigma^{\mu\nu}(1+\gamma_5)(T_a)_{\alpha\beta} b_\beta G^a_{\mu\nu}. \tag{10}$$

Here $p = u, c$ and d denotes down-type quarks, $G^a_{\mu\nu}$ is the chromomagnetic field strength tensor, and T_a are the color $SU(3)$ generators.

The process of obtaining the gauge-invariant operators in SCET requires two-step matching [19,20] because the SCET involves two different scales $\mu \sim m_b$ and $\mu_0 \sim \sqrt{m_b \Lambda}$. First we match the full theory onto $\mathrm{SCET_I}$ at $\mu = m_b$, and we match successively onto $\mathrm{SCET_{II}}$ at μ_0. A concrete example of the two-step matching was illustrated in Ref. [21]. In order to construct the operators in SCET, we first have to specify which quark or antiquark goes to a certain direction to form a light meson. We set $\overline{n}^\mu$ as the direction of a quark-antiquark pair to form a light meson, and n^μ as the direction of the remaining quark which combines with a spectator quark in a B meson to form another meson. Therefore the construction of the operators is process-dependent in the sense that we first specify the direction of each outgoing quark, and the number of operators in SCET is doubled because we have two possibilities to assign two quark fields in both directions.

A generic four-quark operator for nonleptonic B decays in SCET has the form $(\overline{\xi}\Gamma_1 h)(\overline{\chi}\Gamma_2\chi)$, or $(\overline{\chi}\Gamma_1 h)(\overline{\xi}\Gamma_2\chi)$, where Γ_1 and Γ_2 are Dirac matrices, and h is the heavy quark field in the heavy quark effective theory. These operators are derived from the operator $\overline{q}_1\Gamma_1 b \cdot \overline{q}_2\Gamma_2 q_3$ in the full theory where q_i $(i = 1, 2, 3)$ are light quarks. The operator $(\overline{\xi}\Gamma_1 h)(\overline{\chi}\Gamma_2\chi)$ is obtained by replacing q_1 by ξ, and q_2 and q_3 by χ. For the operator $(\overline{\chi}\Gamma_1 h)(\overline{\xi}\Gamma_2\chi)$, we replace q_2 by ξ and q_1 and q_3 by the χ fields. The second operator produces a light meson, in which one quark comes from the heavy-to-light current and another antiquark from the light-to-light current to form a meson in the $\overline{n}^\mu$ direction. A remaining quark goes to the n^μ direction. We can transform this operator to the form $(\overline{\xi}\Gamma_1' h)(\overline{\chi}\Gamma_2'\chi)$ by Fierz transformation. In order to simplify the organization of the computation, all the operators will be written in the form $(\overline{\xi}\Gamma_1 h)(\overline{\chi}\Gamma_2\chi)$. If a Fierz transformation is necessary to obtain this form, we apply Fierz transformation in the full theory and perform matching accordingly. Therefore we only have to consider operators of the form $(\overline{\xi}\Gamma_1 h)(\overline{\chi}\Gamma_2\chi)$, but the two types of the operators with or without Fierz transformation are regarded as distinct and their Wilson coefficients are different.

Though we know the form of the operators, the operator $(\overline{\xi}\Gamma_1 h)(\overline{\chi}\Gamma_2\chi)$ itself is not gauge invariant under the collinear, soft and usoft gauge transformations. In order to obtain gauge-invariant operators in $\mathrm{SCET_{II}}$, we employ two-step matching. We consider the emissions of collinear gluons A^μ_n and $A^\mu_{\overline{n}}$ from external fermions in the full theory and integrate out the off-shell intermediate states of order $\sim m_b$ to obtain the collinear gauge-invariant operators in

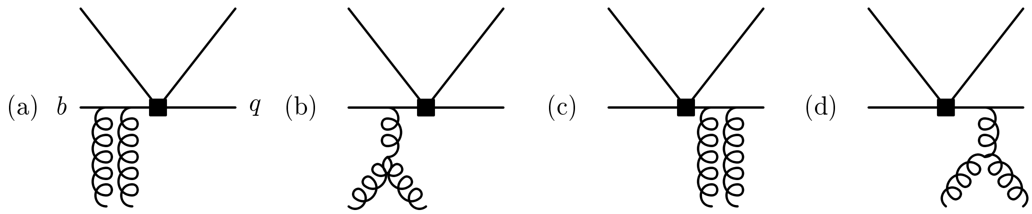

Fig. 1. QCD diagrams attaching two gluons to external fermions to integrate out off-shell modes. The external gluons A_n, $A_{\overline{n}}$ or A_s make the intermediate states off the mass shell. Diagrams with gluons attached to other fermions are omitted.

$\mathrm{SCET_I}$. Then we go down to $\mathrm{SCET_{II}}$, in which we consider the emissions of soft gluons A_s^μ which have momenta of order $\sim m_b\lambda' = m_b\lambda^2$. The emission of soft gluons can cause intermediate states off the mass shell by the amount $p^2 \sim m_b\Lambda$, which should be integrated out to obtain the final form of the operators in $\mathrm{SCET_{II}}$. To simplify the calculation, we will directly match the full-QCD operators to the operators in $\mathrm{SCET_{II}}$ by considering the emissions of the collinear and the soft gluons from each fermion in the full QCD and integrate out all the intermediate states of order $\sim m_b$ and $\sqrt{m_b\Lambda}$ to obtain the gauge-invariant operators in $\mathrm{SCET_{II}}$, as was done in Ref. [15]. The result is equivalent to the two-step matching described above. We will use the two-step matching explicitly in treating the time-ordered products.

We can construct gauge-invariant operators by attaching collinear or soft gluons to each fermion in the full QCD and integrate out off-shell modes. Note that the soft gluons here have momenta of order $\sim m_b\lambda' = m_b\lambda^2$, defined in $\mathrm{SCET_{II}}$. Because the ordering of gauge fields is important due to the non-abelian nature of the gauge fields, we consider corrections to order g^2. Typical Feynman diagrams at order g^2 are shown in Fig. 1, and the remaining diagrams in which gluons are attached to other fermions are omitted. But it is straightforward to attach two gluons to all the fermion lines making intermediate states off-shell.

If collinear and soft gluons are attached to the heavy quark [Fig. 1 (a), (b)], the intermediate heavy quark and gluon states are off the mass shell and we integrate them out. For the collinear quark ξ, the interaction with $A_{\overline{n}}^\mu$or A_s^μ makes the collinear quark off the mass shell. And the interaction with A_n^μ or A_s^μ makes the χ field off the mass shell. We add all the possible configurations in which the intermediate states become off the mass shell and collect the terms at leading order in Λ.

Let us introduce the factors

$$A = \frac{\overline{n}\cdot A_n}{\overline{n}\cdot q_n},\ B = \frac{n\cdot A_{\overline{n}}}{n\cdot q_{\overline{n}}},\ C = \frac{\overline{n}\cdot A_s}{\overline{n}\cdot q_s},\ D = \frac{n\cdot A_s}{n\cdot q_s}, \tag{11}$$

where q_n^μ ($q_{\overline{n}}^\mu$) is the momentum of the collinear gluon A_n^μ ($A_{\overline{n}}^\mu$), and q_s^μ is the soft momentum of the soft gluon A_s^μ. The Wilson lines corresponding to each type of gluons are obtained by exponentiating the above factors as

$$\begin{aligned} W &= \sum_{\rm perm} \exp\Big[-g\frac{1}{\overline{n}\cdot\mathcal{P}}\overline{n}\cdot A_n\Big], \quad \overline{W} = \sum_{\rm perm} \exp\Big[-g\frac{1}{n\cdot\mathcal{Q}}n\cdot A_{\overline{n}}\Big], \\ \overline{S} &= \sum_{\rm perm} \exp\Big[-g\frac{1}{\overline{n}\cdot\mathcal{R}}\overline{n}\cdot A_s\Big], \quad S = \sum_{\rm perm} \exp\Big[-g\frac{1}{n\cdot\mathcal{R}}n\cdot A_s\Big]. \end{aligned} \tag{12}$$

Here $\mathcal{P}^\mu = \overline{n}\cdot\mathcal{P}n^\mu/2 + \mathcal{P}_\perp^\mu$ ($\mathcal{Q}^\mu = n\cdot\mathcal{Q}\overline{n}^\mu/2 + \mathcal{Q}_\perp^\mu$) is the label momentum operator for collinear fields in the n^μ ($\overline{n}^\mu$) direction, and the operator $\mathcal{R}$ is the operator extracting the soft momentum from soft fields.

When we sum over all these diagrams, the color singlet operators of the form $(\overline{\xi}_\alpha\Gamma_1 h_\alpha)(\overline{\chi}_\beta\Gamma_2\chi_\beta)$ and the nonsinglet operators of the form $(\overline{\xi}_\beta\Gamma_1 h_\alpha)(\overline{\chi}_\alpha\Gamma_2\chi_\beta)$ are affected by the gauge fields differently and the final form of the four-quark operators can be written as

$$\begin{aligned} O_S &= (\overline{\xi}_\alpha\Gamma_1 h_\alpha)(\overline{\chi}_\beta\Gamma_2\chi_\beta) \to H^S_{\alpha\beta}L^S_{\gamma\delta}(\overline{\xi}_\alpha\Gamma_1 h_\beta)(\overline{\chi}_\gamma\Gamma_2\chi_\delta), \\ O_N &= (\overline{\xi}_\beta\Gamma_1 h_\alpha)(\overline{\chi}_\alpha\Gamma_2\chi_\beta) \to H^N_{\gamma\beta}L^N_{\alpha\delta}(\overline{\xi}_\alpha\Gamma_1 h_\beta)(\overline{\chi}_\gamma\Gamma_2\chi_\delta), \end{aligned} \tag{13}$$

where $H^O_{\alpha\beta}$, $L^O_{\gamma\delta}$ ($O = S, N$) are the color factors.

One interesting case arises when we attach A_n^μ and $A_{\overline{n}}^\mu$ to a heavy quark. At leading order in Λ, the amplitude in Fig. 1 (a), with A_n^μ and $A_{\overline{n}}^\mu$ carrying the incoming momentum q_n^μ and $q_{\overline{n}}^\mu$ respectively, is given by

$$\begin{aligned} M_a &= \frac{g^2}{m_b(\overline{n}\cdot q_n + n\cdot q_{\overline{n}}) + \overline{n}\cdot q_n n\cdot q_{\overline{n}}} \\ &\times\overline{q}\Gamma_1\Big[\Big(m_b n\cdot A_{\overline{n}} + \overline{n}\cdot q_n n\cdot A_{\overline{n}}\frac{\not{n}\not{\overline{n}}}{4}\Big)\frac{\overline{n}\cdot A_n}{\overline{n}\cdot q_n} \\ &+\Big(m_b\overline{n}\cdot A_n + n\cdot q_{\overline{n}}\overline{n}\cdot A_n\frac{\not{\overline{n}}\not{n}}{4}\Big)\frac{n\cdot A_{\overline{n}}}{n\cdot q_{\overline{n}}}\Big]b, \end{aligned} \tag{14}$$

and the amplitude for Fig. 1 (b) with A_n^μ and $A_{\overline{n}}^\mu$ is written as

$$\begin{aligned} M_b &= \frac{ig^2}{2}f_{abc}\overline{q}\Gamma_1 T_a\frac{n\cdot A^b_{\overline{n}}\overline{n}\cdot A^c_n}{n\cdot q_{\overline{n}}\overline{n}\cdot q_n}b - \frac{ig^2 f_{abc}}{m_b(\overline{n}\cdot q_n + n\cdot q_{\overline{n}}) + \overline{n}\cdot q_n n\cdot q_{\overline{n}}} \\ &\times\overline{q}\Gamma_1 T^a n\cdot A^b_{\overline{n}}\frac{\overline{n}\cdot A^c_n}{\overline{n}\cdot q_n}\Big(m_b + \overline{n}\cdot q_n\frac{\not{n}\not{\overline{n}}}{4}\Big)b. \end{aligned} \tag{15}$$

At first sight, these amplitudes contain complicated denominators which cannot be expressed in terms of A, B, C or D. However, if we add M_a and M_b, we obtain

$$M_a + M_b = \frac{ig^2}{2} f_{abc} \overline{q} \Gamma_1 T_a \frac{n\cdot A_{\overline{n}}^b \overline{n}\cdot A_n^c}{n\cdot q_{\overline{n}} \overline{n}\cdot q_n} b + g^2 \overline{q}\Gamma_1 \frac{\overline{n}\cdot A_n}{\overline{n}\cdot q_n}\frac{n\cdot A_{\overline{n}}}{n\cdot q_{\overline{n}}} b, \tag{16}$$

where there appear only A and B. Therefore the role of the triple gluon vertex is critical not only in determining the order of the Wilson lines but also in making the final expression simple. We will derive gauge-invariant operators using the auxiliary fields in Appendix A, and this cancellation is important in constructing the Lagrangian for the auxiliary fields.

For singlet operators, when we sum over all the Feynman diagrams, we have

$$H_{\alpha\beta}^S = \Big[g(-A+D) - g^2 AD\Big]_{\alpha\beta}, \ L_{\gamma\delta}^S = \delta_{\gamma\delta}, \tag{17}$$

to order g^2. Here we show only the products of two different gauge fields, since they indicate the ordering of the Wilson lines. From Eq. (17), $H_{\alpha\beta}^S$ suggests the Wilson lines to be $(WS^\dagger)_{\alpha\beta}$. For color nonsiglet operators, we have

$$\begin{aligned} H_{\gamma\beta}^N &= \Big[g(-B+C) - g^2 BC\Big]_{\gamma\beta}, \\ L_{\alpha\delta}^N &= \Big[g(-A+B-C+D) \\ &\quad +g^2(AC+DB-CB-DC-AB-AD)\Big]_{\alpha\delta}, \end{aligned} \tag{18}$$

where $H_{\gamma\beta}^N$ suggests the Wilson line in the order $(\overline{W}\overline{S}^\dagger)_{\gamma\beta}$, and $L_{\alpha\delta}^N$ suggests the form $(WS^\dagger \overline{S}\overline{W}^\dagger)_{\alpha\delta}$. Therefore the gauge-invariant singlet and nonsinglet operators in $\mathrm{SCET_{II}}$ are given by

$$\begin{aligned} O_S &= \Big[(\overline{\xi}W)_\alpha \Gamma_1 (S^\dagger h)_\alpha\Big]\cdot\Big[(\overline{\chi}\overline{W})_\beta \Gamma_2 (\overline{W}^\dagger \chi)_\beta\Big], \\ O_N &= \Big[(\overline{\xi}WS^\dagger\overline{S})_\beta \Gamma_1 (\overline{S}^\dagger h)_\alpha\Big]\cdot\Big[(\overline{\chi}\overline{W})_\alpha \Gamma_2 (\overline{W}^\dagger \chi)_\beta\Big]. \end{aligned} \tag{19}$$

The operator O_N can be written as $\Big((\overline{\xi}WS^\dagger)_\beta \Gamma_1 (\overline{S}^\dagger h)_\alpha\Big)\Big((\overline{\chi}\overline{W})_\alpha \Gamma_2 (\overline{S}\overline{W}^\dagger \chi)_\beta\Big)$, but this is identical to O_N since $(WS^\dagger\overline{S})_{\alpha\gamma}\otimes(\overline{W}^\dagger)_{\gamma\beta} = (WS^\dagger)_{\alpha\gamma}\otimes(\overline{S}\overline{W}^\dagger)_{\gamma\beta}$.

All the four-quark operators for B decays are of the form O_S or O_N with different Dirac structure. And the form of the operators O_S and O_N in Eq. (19) manifestly shows the factorization of four-quark operators at leading order in SCET and to all orders in α_s. In the operators O_S and O_N, the interactions of A_n^μ and A_s^μ occur only in the heavy-to-light current sector, while the interactions of $A_{\overline{n}}^\mu$ occur only in the light-to-light current sector. If a collinear or a soft gluon is emitted from one sector and is absorbed by the other sector, the interaction vanishes, or if it does not vanish, the momentum transfer is of order $\sqrt{m_b\Lambda}$, which is already integrated out to produce the Wilson coefficients or jet functions. That is, the gluon exchange between the two current sectors is not allowed, and the gluon exchange is allowed only in each sector. Due to this property, the form of the operators is preserved even though there are any possible exchange of gluons to all orders. The Wilson coefficients may be different at different orders of the perturbation theory. This is an explicit proof of color transparency in SCET, and we can safely calculate the matrix elements of the operators in terms of a product of the matrix elements of the two currents.

Note that the terminology "factorization" is used in two ways. First, it means that the matrix elements of the four-quark operators reduce to products of the matrix elements of the currents, and the matrix element can be written as

$$\langle M_1 M_2 | j_1 \otimes j_2 | B\rangle = \langle M_1 | j_1 | B\rangle \langle M_2 | j_2 | 0\rangle. \tag{20}$$

This was first assumed in naive factorization. It is possible only when there is no gluon exchange which connects the two currents and it is explicitly shown in SCET to all orders in α_s.

Another meaning of factorization appears in high-energy processes in which a physical amplitude can be separated into a short-distance part and a long-distance part. For example, exclusive hadronic form factors at momentum transfer $Q^2 \gg \Lambda^2$ factor into nonperturbative light-cone wave functions ϕ for mesons, convoluted with a hard scattering kernel T as [22]

$$F(Q^2) = \frac{f_a f_b}{Q^2} \int dx dy T(x,y,\mu)\phi_a(x,\mu)\phi_b(y,\mu) + \cdots. \tag{21}$$

We discuss both types of factorization in this paper. So far, we have considered the factorization of matrix elements. The second meaning of factorization will be discussed when we consider spectator contributions.

By matching the four-quark operators in the full theory onto SCET, the effective Hamiltonian for B decays in SCET can be written as

$$\begin{aligned} H_{\mathrm{eff}} = \frac{G_F}{\sqrt{2}} \sum_{T=R,C} \Big[& V_{ub} V_{ud}^* \Big(C_{1T} O_{1T} + C_{2T} O_{2T} \Big) \\ & + \sum_{p=u,c} V_{pb} V_{pd}^* \sum_{i=3,\cdots,6} C_{iT}^p O_{iT} \Big) \Big], \end{aligned} \tag{22}$$

where

$$
\begin{aligned}
O_{1R} &= \Big[(\overline{\xi}^u W)_\alpha (S^\dagger h)_\alpha\Big]_{V-A} \Big[(\overline{\chi}^d \overline{W})_\beta (\overline{W}^\dagger \chi^u)_\beta\Big]_{V-A}, \\
O_{2R} &= \Big[(\overline{\xi}^u W S^\dagger \overline{S})_\beta (\overline{S}^\dagger h)_\alpha\Big]_{V-A} \Big[(\overline{\chi}^d \overline{W})_\alpha (\overline{W}^\dagger \chi^u)_\beta\Big]_{V-A}, \\
O_{3R} &= \Big[(\overline{\xi}^d W)_\alpha (S^\dagger h)_\alpha\Big]_{V-A} \sum_q \Big[(\overline{\chi}^q \overline{W})_\beta (\overline{W}^\dagger \chi^q)_\beta\Big]_{V-A}, \\
O_{4R} &= \Big[(\overline{\xi}^d W S^\dagger \overline{S})_\beta (\overline{S}^\dagger h)_\alpha\Big]_{V-A} \sum_q \Big[(\overline{\chi}^q \overline{W})_\alpha (\overline{W}^\dagger \chi^q)_\beta\Big]_{V-A}, \\
O_{5R} &= \Big[(\overline{\xi}^d W)_\alpha (S^\dagger h)_\alpha\Big]_{V-A} \sum_q \Big[(\overline{\chi}^q \overline{W})_\beta (\overline{W}^\dagger \chi^q)_\beta\Big]_{V+A}, \\
O_{6R} &= \Big[(\overline{\xi}^d W S^\dagger \overline{S})_\beta (\overline{S}^\dagger h)_\alpha\Big)]_{V-A} \sum_q \Big[(\overline{\chi}^q \overline{W})_\alpha (\overline{W}^\dagger \chi^q)_\beta\Big]_{V+A}, \\
O_{1C} &= \Big[(\overline{\xi}^d W S^\dagger \overline{S})_\beta (\overline{S}^\dagger h)_\alpha\Big]_{V-A} \Big[(\overline{\chi}^u \overline{W})_\alpha (\overline{W}^\dagger \chi^u)_\beta\Big]_{V-A}, \\
O_{2C} &= \Big[(\overline{\xi}^d W)_\alpha (S^\dagger h)_\alpha\Big]_{V-A} \Big[(\overline{\chi}^u \overline{W})_\beta (\overline{W}^\dagger \chi^u)_\beta\Big]_{V-A}, \\
O_{3C} &= \sum_q \Big[(\overline{\xi}^q W S^\dagger \overline{S})_\beta (\overline{S}^\dagger h)_\alpha\Big]_{V-A} \Big[(\overline{\chi}^d \overline{W})_\alpha (\overline{W}^\dagger \chi^q)_\beta\Big]_{V-A}, \\
O_{4C} &= \sum_q \Big[(\overline{\xi}^q W)_\alpha (S^\dagger h)_\alpha\Big]_{V-A} \Big[(\overline{\chi}^d \overline{W})_\beta (\overline{W}^\dagger \chi^q)_\beta\Big]_{V-A}.
\end{aligned} \tag{23}
$$

Here the summation over q goes over to light massless quarks. Since we specify the direction of each quark, we have the original operators O_{iR}, obtained from Eq. (9), in which the light-to-light current forms a meson in the $\overline{n}^\mu$ direction. When a light quark in the heavy-to-light current moves in the $\overline{n}^\mu$ direction and forms a meson with an antiquark in the light-to-light current, we use the Fierz transformation first in the full theory and match the operators.

Note that there are no operators O_{5C} and O_{6C} in SCET, which are obtained by Fierzing O_{5R} and O_{6R}. Neglecting color structure and the Wilson lines, O_{5C} and O_{6C} have the form $-2\overline{\xi}(1+\gamma_5)h\cdot\overline{\chi}(1-\gamma_5)\chi$, which is identically zero at leading order in SCET because

$$
\overline{\chi}(1+\gamma_5)\chi = \overline{\chi}\frac{\not{n}\not{\overline{n}}}{4}(1+\gamma_5)\chi = \overline{\chi}(1+\gamma_5)\frac{\not{n}\not{\overline{n}}}{4}\chi = 0, \tag{24}
$$

since $\not{\overline{n}}\chi = 0$. In the literature [7], the effects of the operators O_{5C} and O_{6C} are sometimes known as chirally-enhanced contributions. Even though the effect is formally suppressed, the numerical values may not be negligible. When the equation of motion for the currents is applied, there is an enhancement factor $m_M^2/m_q m_b$, where m_M is a meson mass and m_q is the current quark mass. But in SCET there is simply no such operator as O_{5C} and O_{6C} at leading order. This contribution is formally suppressed in powers of Λ in SCET, but the coefficient of these operators can be large. In phenomenological applications, we have to know how to treat the chirally-enhanced contributions, but we will not consider them here.

4 Matching and the Wilson coefficients in SCET

The Wilson coefficients of the four-quark operators in SCET can be determined by matching the full theory onto SCET. We require that the matrix elements of an operator in the full theory be equal to the matrix elements of the corresponding operator in SCET. We use the $\overline{\mathrm{MS}}$ scheme with naive dimensional regularization scheme and anticommuting γ_5. All the external particles are on the mass shell.

The radiative corrections for the four-quark operators in the full theory are shown in Fig. 2. As a specific example, the amplitudes for Fig. 2 (a) to (d) for the operator $O_1 = (\overline{d}u)_{V-A}(\overline{u}b)_{V-A}$ are given as

$$
\begin{aligned}
iM_a^{(1)} &= \frac{\alpha_s}{4\pi}(T_a)_{jk}(T_a)_{il}(\overline{d}_j u_k)_{V-A}(\overline{u}_i b_l)_{V-A} \\
&\quad \times \Big[\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{1}{\epsilon^2} + \frac{2}{\epsilon}(\ln\frac{x_1 m_b}{\mu} - 1\Big) - 4 + 2\ln\frac{m_b}{\mu} - 2\ln^2\frac{x_1 m_b}{\mu} \\
&\quad + \frac{2-3x_1}{1-x_1}\ln x_1 - 2\mathrm{Li}_2(1-x_1) - \frac{\pi^2}{12}\Big], \\
iM_b^{(1)} &= \frac{\alpha_s}{4\pi}(T_a)_{jk}(T_a)_{il}(\overline{d}_j u_k)_{V-A}(\overline{u}_i b_l)_{V-A} \\
&\quad \times \Big[-\frac{4}{\epsilon_{\mathrm{UV}}} + \frac{1}{\epsilon^2} - \frac{2}{\epsilon}\Big(\ln\frac{x_2 m_b}{\mu} - 1\Big) - 5 + 4\ln\frac{m_b}{\mu} + 2\ln^2\frac{x_2 m_b}{\mu} \\
&\quad - \frac{2}{1-x_2}\ln x_2 + 2\mathrm{Li}_2(1-x_2) + \frac{\pi^2}{12}\Big],
\end{aligned}
$$

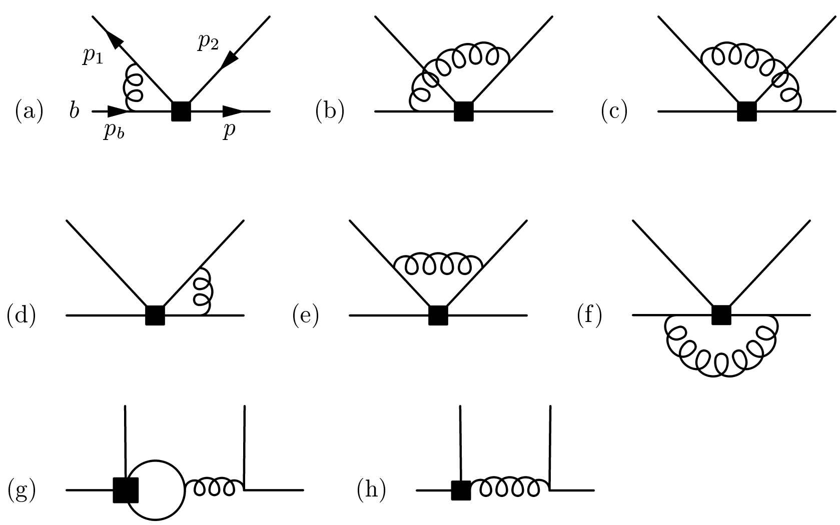

Fig. 2. Radiative corrections at one loop in the full theory. The momenta p_1, p_2, p are outgoing with $p_b = p + p_1 + p_2$. Infrared divergences exist in diagrams (a)–(f). Diagrams (g) and (h) are infrared finite. In (h), the square is the chromomagnetic operator O_8.

$$
\begin{aligned}
iM_c^{(1)} =& \frac{\alpha_s}{4\pi}(T_a)_{jk}(T_a)_{il}(\overline{d}_j u_k)_{V-A}(\overline{u}_i b_l)_{V-A} \\
&\times \Big[-\frac{4}{\epsilon_{\rm UV}} + \frac{2}{\epsilon^2} + \frac{2}{\epsilon}\Big(2 - \ln\frac{-xx_1 m_b^2}{\mu^2}\Big) - 1 + \ln^2\Big(\frac{-xx_1 m_b^2}{\mu^2}\Big) - \frac{\pi^2}{6}\Big], \\
iM_d^{(1)} =& \frac{\alpha_s}{4\pi}(T_a)_{jk}(T_a)_{il}(\overline{d}_j u_k)_{V-A}(\overline{u}_i b_l)_{V-A} \\
&\times \Big[\frac{1}{\epsilon_{\rm UV}} - \frac{2}{\epsilon^2} - \frac{2}{\epsilon}\Big(2 - \ln\frac{-xx_2 m_b^2}{\mu^2}\Big) \\
&-8 + 3\ln\frac{-xx_2 m_b^2}{\mu^2} - \ln^2\frac{-xx_2 m_b^2}{\mu^2} + \frac{\pi^2}{6}\Big],
\end{aligned}
\tag{25}
$$

where $\mathrm{Li}_2(x)$ is the dilogarithmic function. And x, $x_{1,2}$ are the energy fractions given by $x = \overline{n}\cdot p/m_b$, $x_i = n\cdot p_i/m_b$ for $i = 1, 2$. The prescription for $\ln(-x_i)$ is given by $\ln(-x_i - i\epsilon) = \ln x_i - i\pi$. If we add all these "nonfactorizable" contributions, the infrared divergence cancels and the only infrared divergence comes from the vertex corrections of the currents [Fig. 2 (e) and (f)]. Since the vertex correction of the light-to-light current is the same both in the full theory and in SCET, it cancels in matching. And Fig. 2 (f) is given by

$$
\begin{aligned}
iM_f^{(1)} =& \frac{\alpha_s C_F}{4\pi}(\overline{d}u)_{V-A}(\overline{u}b)_{V-A} \\
&\times \Big[\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon^2} + \frac{2}{\epsilon}\Big(\ln\frac{xm_b}{\mu} - 1\Big) - 3 + 2\ln\frac{m_b}{\mu} - 2\ln^2\frac{xm_b}{\mu} \\
&+\frac{2-x}{1-x}\ln x - 2\mathrm{Li}_2(1-x) - \frac{\pi^2}{12}\Big].
\end{aligned}
\tag{26}
$$

For $O_5 = (\overline{d}b)_{V-A}\sum_q(\overline{q}q)_{V+A}$, the corresponding amplitudes are given as

$$
\begin{aligned}
iM_a^{(5)} =& \frac{\alpha_s}{4\pi}(T_a)_{jk}(T_a)_{il}(\overline{d}_i b_l)_{V-A}(\overline{q}_j q_k)_{V+A} \\
&\times \Big[\frac{4}{\epsilon_{\rm UV}} - \frac{1}{\epsilon^2} + \frac{2}{\epsilon}\Big(\ln\frac{x_1 m_b}{\mu} - 1\Big) + 2 - 4\ln\frac{m_b}{\mu} - 2\ln^2\frac{x_1 m_b}{\mu} \\
&+\frac{2}{1-x_1}\ln x_1 - 2\mathrm{Li}_2(1-x_1) - \frac{\pi^2}{12}\Big], \\
iM_b^{(5)} =& \frac{\alpha_s}{4\pi}(T_a)_{jk}(T_a)_{il}(\overline{d}_i b_l)_{V-A}(\overline{q}_j q_k)_{V+A} \\
&\times \Big[-\frac{1}{\epsilon_{\rm UV}} + \frac{1}{\epsilon^2} - \frac{2}{\epsilon}\Big(\ln\frac{x_2 m_b}{\mu} - 1\Big) + 1 - 2\ln\frac{m_b}{\mu} + 2\ln^2\frac{x_2 m_b}{\mu} \\
&-\frac{2-3x_2}{1-x_2}\ln x_2 + 2\mathrm{Li}_2(1-x_2) + \frac{\pi^2}{12}\Big], \\
iM_c^{(5)} =& \frac{\alpha_s}{4\pi}(T_a)_{jk}(T_a)_{il}(\overline{d}_i b_l)_{V-A}(\overline{q}_j q_k)_{V+A}
\end{aligned}
$$

$$\times\Big[-\frac{1}{\epsilon_{\rm UV}}+\frac{2}{\epsilon^2}+\frac{2}{\epsilon}\Big(2-\ln\frac{-xx_1m_b^2}{\mu^2}\Big)+5-3\ln\frac{-xx_1m_b^2}{\mu^2}$$
$$+\ln^2\frac{-xx_1m_b^2}{\mu^2}-\frac{\pi^2}{6}\Big],$$
$$iM_d^{(5)}=\frac{\alpha_s}{4\pi}(T_a)_{jk}(T_a)_{il}(\overline{d}_ib_l)_{V-A}(\overline{q}_jq_k)_{V+A}$$
$$\times\Big[\frac{4}{\epsilon_{\rm UV}}-\frac{2}{\epsilon^2}-\frac{2}{\epsilon}\Big(2-\ln\frac{-xx_2m_b^2}{\mu^2}\Big)-2-\ln^2\frac{-xx_2m_b^2}{\mu^2}+\frac{\pi^2}{6}\Big], \quad (27)$$

and $iM_f^{(5)}$ is the same as $iM_f^{(1)}$ except the Dirac structure. The infrared divergence of the nonfactorizabie contributions in Eq. (27) cancels, and the infrared divergence from the vertex corrections is cancelled in matching.

Using the color identity

$$(T_a)_{jk}(T_a)_{il}=\frac{1}{2}\delta_{jl}\delta_{ik}-\frac{1}{2N}\delta_{jk}\delta_{il}, \quad (28)$$

the operators in Eq. (25) becomes $O_1/2-O_2/(2N)$, and the operators in Eq. (27) becomes $O_5/2-O_6/(2N)$. In the radiative corrections of O_2, the operator becomes C_FO_2 due to the color factors in Fig. 2(a) and (d), while it becomes $O_1/2-O_2/(2N)$ from Fig. 2(b) and (c). Therefore the radiative corrections for O_2, or for nonsinglet operators in general, have infrared divergences in all the diagrams.

The corresponding radiative corrections for the four-quark operators in SCET are shown in Fig. 3. For singlet operators, the radiative corrections exist only in the heavy-to-light current sector [Fig. 3 (a), (b)], in which the infrared

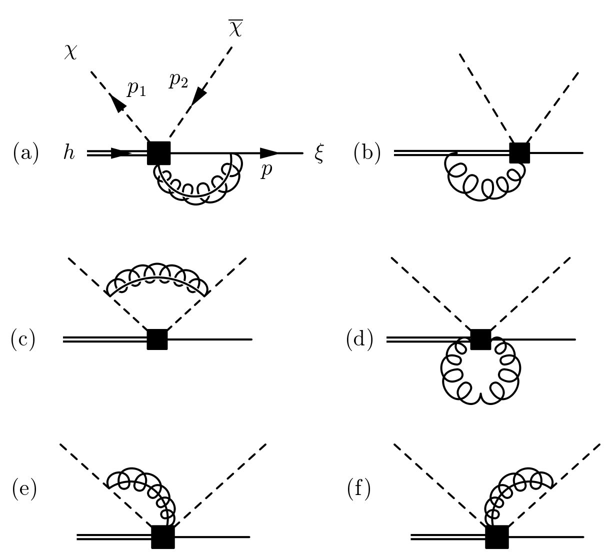

Fig. 3. Radiative corrections at one loop in SCET. Curly lines with a line represent collinear gluons, and curly lines represent soft gluons.

divergence appearing in the calculation is exactly the same as the infrared divergence in the full theory. The diagram (c) is included, but since the result is the same in the full theory, it is cancelled in matching. For nonsinglet operators, we need all the diagrams in Fig. 3 with additional collinear gluon interactions in the light-to-light current sector [Fig. 3 (e) and (f)] and soft gluon interactions [Fig. 3 (b) and (d)]. When we calculate these diagrams, the infrared divergence is exactly the same as that in the full theory. Therefore we can safely match the full theory onto SCET since the infrared divergence cancels, and the Wilson coefficients can be calculated.

In matching the theory at one loop, we calculate perturbative matrix elements in the full and effective theories. All the long-distance physics is reproduced in the effective theory, and the difference between the two calculations determines the short-distance Wilson coefficients. We treat both ultraviolet and infrared divergences using dimensional regularization, with the final collinear quark on the mass shell. In this case all the Feynman diagrams in SCET are proportional to $1/\epsilon_{\rm UV} - 1/\epsilon_{\rm IR} = 0$. The ultraviolet divergences are cancelled by the counterterms in SCET, and all the infrared divergences cancel in the difference between the full theory and SCET. Therefore the Wilson coefficients of various operators in SCET can be obtained by calculating radiative corrections in the full theory. In the full theory we also have to consider the Feynman diagrams with fermion loops and the effect of the chromomagnetic operator. These contributions are included in Fig. 2 (g) and (h).

The Wilson coefficients are, in general, functions of the operators $\overline{n}\cdot\mathcal{P}/m_b$, $n\cdot\mathcal{Q}/m_b$ and $n\cdot\mathcal{Q}^\dagger/m_b$. The matrix elements of the four-quark operators can be written in terms of a convolution as

$$
\begin{aligned}
M = \int_0^1 & dx dx_1 dx_2 C(x,x_1,x_2) \langle \overline{\xi} W \delta\Big(x - \frac{\overline{n}\cdot\mathcal{P}^\dagger}{m_b}\Big)\Gamma_1 S^\dagger h \\
& \times \overline{\chi}\overline{W}\delta\Big(x_1 - \frac{n\cdot\mathcal{Q}^\dagger}{m_b}\Big)\Gamma_2 \delta\Big(x_2 + \frac{n\cdot\mathcal{Q}}{m_b}\Big)\overline{W}^\dagger \chi\rangle
\end{aligned}
\tag{29}
$$

Due to the momentum conservation x, x_1 and x_2 satisfiy $x+x_1+x_2=2$. For nonleptonic decays into two light mesons at leading order in SCET, we can set $x=1$, $x_1=u$, and $x_2=\overline{u}\equiv 1-x_1$, and the matrix element can be written as

$$
M \rightarrow \int d\eta\ C(\eta)\langle \overline{\xi} W \Gamma_1 S^\dagger h \cdot \overline{\chi}\overline{W}\delta(\eta - \mathcal{Q}_+)\Gamma_2 \overline{W}^\dagger \chi\rangle. \tag{30}
$$

where $u=x_1 = 1-x_2 = \eta/(4E)+1/2$ with $\mathcal{Q}_+ = n\cdot\mathcal{Q}^\dagger + n\cdot\mathcal{Q}$. However, we list all the Wilson coefficients for general x, x_1 and x_2, which will be useful for other decay modes, or nonleptonic B decays at subleading order in SCET.

By adding the wave function renormalization of the heavy quark, the Wilson coefficients at next-to-leading order are given as

$$
\begin{aligned}
C_{1R} &= \Big[1-\frac{A_1}{2N}+C_FA_2\Big]C_1+\frac{A_3}{2}C_2,\ \ C_{2R}=\frac{A_1}{2}C_1+\Big[1-\frac{A_3}{2N}+C_FA_4\Big]C_2,\\
C^p_{3R} &= \Big[1-\frac{A_1}{2N}+C_FA_2\Big]C_3+\frac{A_3}{2}C_4,\ \ C^p_{4R}=\frac{A_1}{2}C_3+\Big[1-\frac{A_3}{2N}+C_FA_4\Big]C_4,\\
C^p_{5R} &= \Big[1-\frac{A_5}{2N}+C_FA_2\Big]C_5+\frac{A_6}{2}C_6,\ \ C^p_{6R}=\frac{A_5}{2}C_5+\Big[1-\frac{A_6}{2N}+C_FA_7\Big]C_6,\\
C_{1C} &= \Big[1-\frac{A_3}{2N}+C_FA_4\Big]C_1+\frac{A_1}{2}C_2,\ \ C_{2C}=\frac{A_3}{2}C_1+\Big[1-\frac{A_1}{2N}+C_FA_2\Big]C_2\\
C^p_{3C} &= \Big[1-\frac{A_3}{2N}+C_FA_4\Big]C_3+\frac{A_1}{2}C_4-\frac{1}{2N}C^p_l,\\
C^p_{4C} &= \frac{A_3}{2}C_3+\Big[1-\frac{A_1}{2N}+C_FA_2\Big]C_4+\frac{1}{2}C^p_l,
\end{aligned}
\tag{31}
$$

where C_i's are the Wilson coefficients from the full theory. And the coefficients A_i to order α_s, evaluated at $\mu = m_b$ are given as

$$
\begin{aligned}
A_1 &= \frac{\alpha_s}{4\pi}\Big[-18+3\ln(-x)+\ln\frac{x^2}{x_1x_2}\ln\frac{x_1}{x_2}+\frac{2-3x_1}{1-x_1}\ln x_1+\frac{1-3x_2}{1-x_2}\ln x_2\\
&\quad -2\mathrm{Li}_2(1-x_1)+2\mathrm{Li}_2(1-x_2)\Big]\\
A_2 &= \frac{\alpha_s}{4\pi}\Big[-5-2\ln^2 x+\frac{2-x}{1-x}\ln x-2\mathrm{Li}_2(1-x)-\frac{\pi^2}{12}\Big]\\
A_3 &= \frac{\alpha_s}{4\pi}\Big[-9+\frac{2-x}{1-x}\ln x-2\mathrm{Li}_2(1-x)+2\ln^2 x_2+2\ln^2(-x_1)-\ln^2\frac{-x_1}{x}\\
&\quad -\frac{2}{1-x_2}\ln x_2+2\mathrm{Li}_2(1-x_2)-\frac{\pi^2}{6}\Big],\\
A_4 &= \frac{\alpha_s}{4\pi}\Big[-14-2\ln^2 x_1-\ln^2(-xx_2)+3\ln(-xx_2)+\frac{2-3x_1}{1-x_1}\ln x_1\\
&\quad -2\mathrm{Li}_2(1-x_1)+\frac{\pi^2}{12}\Big],\\
A_5 &= \frac{\alpha_s}{4\pi}\Big[6-3\ln(-x)+\ln\frac{x^2}{x_1x_2}\ln\frac{x_1}{x_2}-\frac{1-3x_1}{1-x_1}\ln x_1-\frac{2-3x_2}{1-x_2}\ln x_2\\
&\quad -2\mathrm{Li}_2(1-x_1)+2\mathrm{Li}_2(1-x_2)\Big],\\
A_6 &= \frac{\alpha_s}{4\pi}\Big[3+2\ln^2 x_2+2\ln^2(-x_1)-\ln^2\frac{-x_1}{x}-\frac{1-2x}{1-x}\ln x-3\ln(-x_1)\\
&\quad -\frac{2-3x_2}{1-x_2}\ln x_2-2\mathrm{Li}_2(1-x)+2\mathrm{Li}_2(1-x_2)-\frac{\pi^2}{6}\Big],\\
A_7 &= \frac{\alpha_s}{4\pi}\Big[-2-2\ln^2 x_1-\ln^2(-xx_2)+\frac{2}{1-x_1}\ln x_1
\end{aligned}
$$

$$-2\mathrm{Li}_2(1-x_1)+\frac{\pi^2}{12}\Big]. \tag{32}$$

And the contribution C_l^p from fermion loops and the chromomagnetic operator [Fig. 2 (g) and (h)] at $\mu = m_b$ are given by

$$\begin{aligned} C_l^p = \frac{\alpha_s}{4\pi}\Big[& C_1\Big(\frac{2}{3}-G(s_p)\Big)+C_3\Big(\frac{4}{3}-G(0)-G(1)\Big) \\ & +C_4\Big(\frac{2n_f}{3}-3G(0)-G(s_c)-G(1)\Big) \\ & +C_6\Big(-3G(0)-G(s_c)-G(1)\Big)-(C_5+C_8)\frac{2}{1-x_1}\Big], \end{aligned} \tag{33}$$

where $s_p = m_p^2/m_b^2$ ($s_c = m_c^2/m_b^2$), and $G(s)$ is given by

$$G(s) = -4\int_0^1 dz\, z(1-z)\Big(s-z(1-z)(1-x_1)-i\epsilon\Big). \tag{34}$$

We can add $C_l^p/(2N)$ in C_3^R and C_5^R, and $C_l^p/2$ in C_4^R and C_6^R with $x_1 \to x$ in Eq. (33). But in physical processes in which the final-state consists of color singlet mesons, the contribution of C_l^p cancels, hence omitted in Eq. (31).

5 Nonfactorizable spectator contributions

In nonleptonic B decays, we also have to consider the spectator quark which can interact with the b quark or other quarks forming light mesons. In this section, we concentrate on the spectator quark interacting with the quark and antiquark pair (χ fields) produced by the weak current, which we call nonfactorizable spectator contributions. The spectator quark can interact with the heavy-to-light current, which will be treated in the next section. Here we apply the two-step matching explicitly. The time-ordered products are constructed in $\mathrm{SCET_I}$, and they are evolved down to $\mathrm{SCET_{II}}$.

Nonfactorizable spectator contributions arise from the interactions of the gluons A_n^μ in the light-to-light current with a spectator quark, which becomes a collinear quark as a result. However the operators O_S and O_N at leading order in SCET do not involve the interaction of A_n^μ. Therefore we take into account subleading operators which involve A_n^μ in the light-to-light current. And the Lagrangian describing the interaction of collinear and usoft quarks begins with $\mathcal{O}(\lambda)$ compared to the leading collinear Lagrangian. But the propagator of the exchanged gluon is of order λ^{-2}. Therefore the nonfactorizable spectator contribution is of the same order as the leading contributions from the four-quark

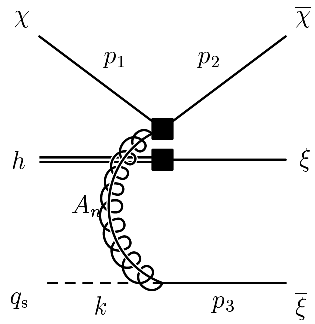

Fig. 4. A Feynman diagram for nonfactorizable spectator contributions from the subleading operator in the light-to-light current. The soft momentum k is incoming, and p_i $(i = 1, 2, 3, 4)$ are outgoing.

operators. In order to evaluate decay amplitudes at leading order in SCET, we need to include the nonfactorizable spectator contributions, as shown in Fig. 4. Here we consider a collinear gluon A_n^μ from the light-to-light current interacting with a spectator quark to produce a collinear quark ξ.

The Lagrangian for the collinear and usoft quarks at order λ is given by [21,23]

$$\mathcal{L}_{\xi q}^{(1)} = \overline{q}_{us}\Big[W^\dagger i\not{\mathcal{D}}_n^\perp W\Big] W^\dagger \xi + \text{h.c.}. \tag{35}$$

In SCET_I, there are two independent subleading operators suppressed by λ, in which collinear gluons A_n^μ come from the current $\overline{\chi}\Gamma_2\chi$. They are given by

$$\begin{aligned} O_i^{(1a)} &= \Big((\overline{\xi}W)_\beta \Gamma_{1i} h_\alpha\Big)\Big\{\Big(\overline{\chi}\overline{W}\frac{1}{n\cdot\mathcal{Q}^\dagger}\frac{\not{n}}{2}\Big[W^\dagger i\overleftarrow{\not{\mathcal{D}}}_{n\perp} W\Big]\Big)_\alpha \Gamma_{2i}(\overline{W}^\dagger\chi)_\beta\Big\} \\ O_i^{(1b)} &= \Big((\overline{\xi}W)_\beta \Gamma_{1i} h_\alpha\Big)\Big\{(\overline{\chi}\overline{W})_\alpha \Gamma_{2i}\Big(\Big[W^\dagger i\overrightarrow{\not{\mathcal{D}}}_{n\perp} W\Big]\frac{\not{n}}{2}\frac{1}{n\cdot\mathcal{Q}}\overline{W}^\dagger\chi\Big)_\beta\Big\}, \end{aligned} \tag{36}$$

where the index i runs over all the possible forms of the operators as shown in Eq. (23). The Wilson coefficients of these operators are 1 at lowest order in α_s. The form of the operators $O_i^{(1a,1b)}$ can be obtained in a straightforward manner, but the ordering of the Wilson lines is nontrivial. It will be explained using the auxiliary field method in Appendix B.

In SCET_I, the nonfactorizable spectator contribution is given by the matrix elements of the time-ordered product

$$T_i^{(1)} = \int d^4x T\Big[O_i^{(1a)}(0) + O_i^{(1b)}(0), i\mathcal{L}_{\xi q}^{(1)}(x)\Big]. \tag{37}$$

In order to go down to SCET_II, we decouple the collinear-usoft interaction using the field redefinitions [15]

$$\xi^{(0)} = Y^\dagger \xi, \quad A_n^{(0)} = Y^\dagger A_n Y, \quad Y(x) = \mathrm{P}\exp\Big(ig\int_{-\infty}^{x} ds n\cdot A_{us}(ns)\Big),$$
$$\chi^{(0)} = \overline{Y}^\dagger \chi, \quad A_{\overline{n}}^{(0)} = \overline{Y}^\dagger A_{\overline{n}} \overline{Y}, \quad \overline{Y}(x) = \mathrm{P}\exp\Big(ig\int_{-\infty}^{x} ds\overline{n}\cdot A_{us}(\overline{n}s)\Big). \qquad (38)$$

The collinear effective Lagrangian can be written in terms of $\xi^{(0)}$, $\chi^{(0)}$ and $A_n^{(0)\mu}$, $A_{\overline{n}}^{(0)\mu}$, that is, the collinear-usoft interactions are factorized with the field redefinitions given in Eq. (38).

Matching at $\mu_0 \sim \sqrt{m_b\Lambda}$, the small expansion parameter changes from $\lambda \sim \sqrt{\Lambda/m_b}$ to $\lambda' \sim \Lambda/m_b$ with a definite power counting procedure for any operators. Recall that we rename the usoft field of order $p \sim m_b\lambda^2 \sim m_b\lambda'$ as the soft field in $\mathrm{SCET_{II}}$. In that sense the usoft fields become soft and the Wilson line Y becomes the Wilson line with soft gluons in $\mathrm{SCET_{II}}$ but “soft” means the momentum of order $m_b\lambda'$. Without introducing addtional Wilson lines, we denote this Wilson line as S and in the matching we replace $Y \to S$ without any ambiguity. And the operators are matched onto the operators in $\mathrm{SCET_{II}}$.

The nonfactorizable spectator contribution comes from the matrix elements of six-quark operators. In calculating the matrix elements of $T_i^{(1)}$, we first factorize the usoft-collinear interactions using Eq. (38) with $Y \to S$, $\overline{Y} \to \overline{S}$, and project out color indices in such a way that the quark bilinears ($\overline{\xi}W$, $W^\dagger\xi$) and ($\overline{\chi}\overline{W}$, $\overline{W}^\dagger\chi$) forming mesons are color singlets. Since the final expression involving the Wilson lines are nontrivial, we show explicitly how the color projection is performed for $O_i^{(1a)}$. At order α_s, we will extract a collinear gluon from the factor $(W^\dagger i\not{D}_{n\perp}W)$ in $\mathcal{L}_{\xi q}$ and $O_i^{(1a)}$ to contract them. Neglecting the Dirac structure and keeping color-dependent parts only, the time-ordered product of $O_i^{(1a)}$ with $\mathcal{L}_{\xi q}$ has the form

$$\begin{aligned}
& \Big[(\overline{\xi}WS^\dagger)_\beta\cdot h_\alpha\Big]\Big[\Big(\overline{\chi}\overline{W}\overline{S}^\dagger ST_aS^\dagger\Big)_\alpha\cdot(\overline{S}\overline{W}^\dagger\chi)_\beta\Big]\Big[(\overline{q}_sS)_\gamma(T_a)_{\gamma\delta}(W^\dagger\xi)_\delta\Big] \\
= \;& \Big[(\overline{\xi}W)_\mu(S^\dagger)_{\mu\beta}\cdot h_\alpha\Big]\Big[(\overline{\chi}\overline{W})_\nu\Big(\overline{S}^\dagger ST_aS^\dagger\Big)_{\nu\alpha}\cdot(\overline{S})_{\beta\rho}(\overline{W}^\dagger\chi)_\rho\Big] \\
& \times\Big[(\overline{q}_sS)_\gamma(T_a)_{\gamma\delta}(W^\dagger\xi)_\delta\Big] \\
= \;& \Big[(\overline{\xi}W)_\mu\cdot h_\alpha\Big]\Big[(\overline{\chi}\overline{W})_\nu\cdot(\overline{W}^\dagger\chi)_\rho\Big]\Big[(\overline{q}_sS)_\gamma\cdot(W^\dagger\xi)_\delta\Big] \\
& \times\Big(\overline{S}^\dagger ST_aS^\dagger\Big)_{\nu\alpha}(S^\dagger\overline{S})_{\mu\rho}(T_a)_{\gamma\delta} \\
\longrightarrow \;& \Big[(\overline{\xi}W)_\beta\cdot h_\alpha\Big]\Big[(\overline{\chi}\overline{W})_\nu\cdot(\overline{W}^\dagger\chi)_\nu\Big]\Big[(\overline{q}_sS)_\gamma\cdot(W^\dagger\xi)_\beta\Big] \\
& \times\frac{1}{N^2}(S^\dagger\overline{S})_{\mu\rho}(\overline{S}^\dagger ST_aS^\dagger)_{\rho\alpha}(T_a)_{\gamma\mu} \\
= \;& \frac{C_F}{N^2}\Big[(\overline{\xi}W)_\beta\cdot(S^\dagger h)_\alpha\Big]\Big[(\overline{\chi}\overline{W})_\nu\cdot(\overline{W}^\dagger\chi)_\nu\Big]\Big[(\overline{q}_sS)_\alpha\cdot(W^\dagger\xi)_\beta\Big],
\end{aligned} \qquad (39)$$

where the dots denote the Dirac structure and the color projection is performed after the arrow in Eq. (39). The ime-ordered product with $O_i^{(1b)}$ can be projected in the same way. The time-ordered products of the operators $O_i^{(1a,b)}$ with the color singlet structure vanish due to the color structure.

The matrix element of the time-ordered product $\langle T_i^{(1)}\rangle$ at order α_s is given by

$$
\begin{aligned}
\langle T_i^{(1)}\rangle =& -4\pi\alpha_s \frac{C_F}{N^2}\int d\overline{n}\cdot x \int \frac{dn\cdot k}{4\pi} e^{in\cdot k\overline{n}\cdot x/2} \\
&\times \Big\{ \langle \Big[(\overline{\xi}W)_\beta \Gamma_{1i}(S^\dagger h)_\alpha\Big]\Big[(\overline{\chi}\overline{W})_\gamma \frac{1}{n\cdot\mathcal{Q}^\dagger}\frac{\not{n}}{2}\gamma_\perp^\mu \Gamma_{2i}(\overline{W}^\dagger\chi)_\gamma\Big] \\
&\quad \times\Big[(\overline{q}_s S)_\alpha(\overline{n}\cdot x)\frac{1}{n\cdot\mathcal{R}^\dagger}\gamma_\mu^\perp \frac{1}{\overline{n}\cdot\mathcal{P}}(W^\dagger\xi)_\beta(0)\Big]\rangle \\
&+\langle \Big[(\overline{\xi}W)_\beta \Gamma_{1i}(S^\dagger h)_\alpha\Big]\Big[(\overline{\chi}\overline{W})_\gamma \Gamma_{2i}\gamma_\perp^\mu \frac{\not{n}}{2}\frac{1}{n\cdot\mathcal{Q}}(\overline{W}^\dagger\chi)_\gamma\Big] \\
&\quad \times\Big[(\overline{q}_s S)_\alpha(\overline{n}\cdot x)\frac{1}{n\cdot\mathcal{R}^\dagger}\gamma_\mu^\perp \frac{1}{\overline{n}\cdot\mathcal{P}}(W^\dagger\xi)_\beta(0)\Big]\rangle \Big\}. \qquad (40)
\end{aligned}
$$

Let us consider in detail the matrix elements in Eq. (40) for different Dirac structure $\Gamma_{1i}\otimes\Gamma_{2i}$. For simplicity we omit all the Wilson lines and the momentum operators in the following calculation. For $\gamma_\nu(1-\gamma_5)\otimes\gamma^\nu(1-\gamma_5)$, we can evaluate the first term in the curly bracket in Eq. (40) as

$$
\begin{aligned}
&\langle \overline{\xi}_\beta \gamma_\nu(1-\gamma_5)h_\alpha\cdot\overline{\chi}\frac{\not{n}}{2}\gamma_\perp^\mu\gamma^\nu(1-\gamma_5)\chi\cdot\overline{q}_\alpha\gamma_\mu^\perp\xi_\beta\rangle \\
&=\langle \overline{\xi}_\beta \gamma_\nu^\perp(1-\gamma_5)h_\alpha\cdot\overline{\chi}\frac{\not{n}}{2}(2g_\perp^{\mu\nu}-\gamma_\perp^\nu\gamma_\perp^\mu)(1-\gamma_5)\chi\cdot\overline{q}_\alpha\gamma_\mu^\perp\xi_\beta\rangle \\
&=2\langle \overline{\xi}_\beta \gamma_\perp^\mu(1-\gamma_5)h_\alpha\cdot\overline{q}_\alpha\gamma_\mu^\perp\xi_\beta\cdot\overline{\chi}\frac{\not{n}}{2}(1-\gamma_5)\chi\rangle \\
&-\langle \overline{\xi}_\beta \gamma_\nu^\perp(1-\gamma_5)h_\alpha\cdot\overline{\chi}\frac{\not{n}}{2}\gamma_\perp^\nu\gamma_\perp^\mu(1-\gamma_5)\chi\cdot\overline{q}_\alpha\gamma_\mu^\perp\xi_\beta\rangle. \qquad (41)
\end{aligned}
$$

In the first line we can replace γ_ν by $\gamma_\nu^\perp$ and we use the Fierz transformation to arrive at the last relation. The first term in the last relation in Eq. (41) can be further simplified as

$$
\begin{aligned}
&2\overline{\xi}_\beta\gamma_\perp^\mu(1-\gamma_5)h_\alpha\cdot\overline{q}_\alpha\gamma_\mu^\perp\xi_\beta\cdot\overline{\chi}\frac{\not{n}}{2}(1-\gamma_5)\chi \\
&=\frac{1}{2}\overline{\xi}_\beta\gamma^\mu(1-\gamma_5)h_\alpha\cdot\overline{q}_\alpha\Big[\gamma_\mu(1-\gamma_5)+\gamma_\mu(1+\gamma_5)\Big]\xi_\beta\cdot\overline{\chi}\not{n}(1-\gamma_5)\chi \\
&=\frac{1}{2}\overline{\xi}\gamma^\mu(1-\gamma_5)\xi\cdot\overline{q}\gamma_\mu(1-\gamma_5)h\cdot\overline{\chi}\not{n}(1-\gamma_5)\chi
\end{aligned}
$$

$$= \frac{1}{4}\overline{\xi}\overline{n}\!\!\!/(1-\gamma_5)\xi \cdot \overline{q}n\!\!\!/(1-\gamma_5)h \cdot \overline{\chi}n\!\!\!/(1-\gamma_5)\chi. \tag{42}$$

The part proportional to $\gamma^\mu(1+\gamma_5)$ vanishes when we apply the Fierz transformation. In the third line we use the Fierz transformation for the product of the first two currents. Similarly, the second term in Eq. (41) is simplified as

$$\begin{aligned}
&-\overline{\xi}_\beta\gamma_\nu(1-\gamma_5)h_\alpha \cdot \overline{\chi}\frac{\overline{n}\!\!\!/}{2}\gamma_\perp^\nu\gamma_\perp^\mu(1-\gamma_5)\chi \cdot \overline{q}_\alpha\gamma_\mu^\perp\xi_\beta \\
&= -\overline{\xi}_\beta\gamma_\nu(1-\gamma_5)h_\alpha \cdot \overline{\chi}\frac{\overline{n}\!\!\!/}{2}\gamma^\nu(1+\gamma_5)\gamma_\perp^\mu\chi \cdot \overline{q}_\alpha\gamma_\mu^\perp\xi_\beta \\
&= 2\overline{\xi}_\beta\gamma_\perp^\mu(1-\gamma_5)\chi_\gamma \cdot \overline{\chi}_\gamma\frac{\overline{n}\!\!\!/}{2}(1-\gamma_5)h_\alpha \cdot \overline{q}_\alpha\gamma_\mu^\perp\xi_\beta \\
&= \overline{\xi}_\beta\gamma^\mu(1-\gamma_5)\chi_\gamma \cdot \overline{q}_\alpha\gamma_\mu(1-\gamma_5)\xi_\beta \cdot \overline{\chi}_\gamma\frac{\overline{n}\!\!\!/}{2}(1-\gamma_5)h_\alpha \\
&= \frac{1}{2}\overline{\xi}\overline{n}\!\!\!/(1-\gamma_5)\xi \cdot \overline{q}_\alpha n\!\!\!/(1-\gamma_5)\chi_\gamma \cdot \overline{\chi}_\gamma(1+\gamma_5)\frac{\overline{n}\!\!\!/}{2}h_\alpha \\
&= -\frac{1}{4}\overline{\xi}\overline{n}\!\!\!/(1-\gamma_5)\xi \cdot \overline{q}n\!\!\!/(1-\gamma_5)h \cdot \overline{\chi}n\!\!\!/(1-\gamma_5)\chi,
\end{aligned} \tag{43}$$

where the Fierz transformations are applied successively. When we add Eqs. (42) and (43), they cancel. The second term in Eq. (40) is given by Eq. (43) with an opposite sign. As a result, for $\Gamma_{1i}\otimes\Gamma_{2i} = \gamma_\nu(1-\gamma_5)\otimes\gamma^\nu(1-\gamma_5)$, only the second term in Eq. (40) contributes and the matrix element is given as

$$\begin{aligned}
&\frac{1}{4}\langle\overline{\xi}W\frac{1}{\overline{n}\cdot\mathcal{P}}W^\dagger\overline{n}\!\!\!/(1-\gamma_5)\xi\rangle\langle\overline{q}_s S\frac{1}{n\cdot\mathcal{R}^\dagger}S^\dagger n\!\!\!/(1-\gamma_5)h\rangle \\
&\times\langle\overline{\chi}\overline{W}\frac{1}{n\cdot\mathcal{Q}}\overline{W}^\dagger n\!\!\!/(1-\gamma_5)\chi\rangle.
\end{aligned} \tag{44}$$

Note that the matrix element of the six-quark operators in Eq. (44) is written in a factorized form. It is because a collinear or a soft gluon in one sector cannot interact with the particles in the other sector. That type of interaction makes the intermediate states off the mass shell by $\sim m_b$ or $\sqrt{m_b\Lambda}$. These off-shell modes are already taken care of in the Wilson coefficients or in the jet functions. Therefore the six-quark operators describing the spectator interactions are factorized and the matrix elements of the six-quark operators can be computed by the products of the matrix elements of each current.

For $\gamma_\nu(1-\gamma_5)\otimes\gamma^\nu(1+\gamma_5)$, only the first term in Eq. (40) contributes and the matrix element is given by

$$\frac{1}{4}\langle\overline{\xi}W\frac{1}{\overline{n}\cdot\mathcal{P}}W^\dagger\overline{n}\!\!\!/(1-\gamma_5)\xi\rangle\langle\overline{q}_s S\frac{1}{n\cdot\mathcal{R}^\dagger}S^\dagger n\!\!\!/(1-\gamma_5)h\rangle$$

$$\times\langle\overline{\chi}\overline{W}\frac{1}{n\cdot\mathcal{Q}^\dagger}\overline{W}^\dagger\not{n}(1+\gamma_5)\chi\rangle. \tag{45}$$

And for $(1-\gamma_5)\otimes(1+\gamma_5)$, the matrix element is given as

$$\frac{1}{16}\langle\overline{\xi}W\frac{1}{\overline{n}\cdot\mathcal{P}}W^\dagger\not{\overline{n}}(1+\gamma_5)\xi\rangle\langle\overline{q}_sS\frac{1}{n\cdot\mathcal{R}^\dagger}S^\dagger\not{n}\gamma_\perp^\mu(1-\gamma_5)h\rangle$$
$$\times\langle\overline{\chi}\overline{W}\Big(\frac{1}{n\cdot\mathcal{Q}^\dagger}-\frac{1}{n\cdot\mathcal{Q}}\Big)\overline{W}^\dagger\not{n}\gamma_\mu^\perp(1+\gamma_5)\chi\rangle. \tag{46}$$

As can be clearly seen in the final expressions in Eqs. (44)–(46), the gluons A_n^μ, $A_{\overline{n}}^\mu$, and A_s^μ can be exchanged only inside each meson. This is true at higher orders of α_s. Though the Wilson coefficients can be different, the structure of the operator is the same to all orders in α_s. Therefore the nonfactorizable spectator contribution is factorized to all orders in α_s.

Let us calculate the matrix element $T_i^{(1)}$ explicitly for $\Gamma_{1i}\otimes\Gamma_{2i}=\gamma_\nu(1-\gamma_5)\otimes\gamma^\nu(1-\gamma_5)$. It is given by

$$\begin{aligned}\langle T_i^{(1)}\rangle &= -4\pi\alpha_s\frac{C_F}{N^2}\int d\overline{n}\cdot x\int\frac{dn\cdot k}{4\pi}\frac{e^{in\cdot k\overline{n}\cdot x/2}}{n\cdot kn\cdot p_2\overline{n}\cdot p_3}\\ &\quad\times\frac{1}{4}\langle M_1|\overline{\xi}W\not{\overline{n}}(1-\gamma_5)W^\dagger\xi|0\rangle\langle M_2|\overline{\chi}\overline{W}\not{n}(1-\gamma_5)\overline{W}^\dagger\chi|0\rangle\\ &\quad\times\langle 0|\overline{q}_sS(\overline{n}\cdot x)\not{n}(1-\gamma_5)S^\dagger h(0)|B\rangle,\end{aligned} \tag{47}$$

where we integrate out $n\cdot x$ and $x_\perp$ explicitly. The matrix element involving the B meson can be calculated as

$$\begin{aligned}&\langle 0|\overline{q}_sS(\overline{n}\cdot x)\not{n}(1-\gamma_5)S^\dagger h|B\rangle=\int dr_+e^{-ir_+\overline{n}\cdot x/2}\mathrm{Tr}\,\Big[\Psi_B(r_+)\not{n}(1-\gamma_5)\Big]\\ &=-\frac{if_Bm_B}{4}\int dr_+e^{-ir_+\overline{n}\cdot x/2}\mathrm{Tr}\,\Big[\frac{1+\not{v}}{2}\not{\overline{n}}\gamma_5\not{n}(1-\gamma_5)\Big]\phi_B^+(r_+)\\ &=-if_Bm_B\int dr_+e^{-ir_+\overline{n}\cdot x/2}\phi_B^+(r_+),\end{aligned} \tag{48}$$

where the leading-twist B meson light-cone wave function is defined through the projection of the B meson as [26,27]

$$\Psi_B(r_+)=-\frac{if_Bm_B}{4}\Big[\frac{1+\not{v}}{2}\Big(\not{\overline{n}}\phi_B^+(r_+)+\not{n}\phi_B^-(r_+)\Big)\gamma_5\Big]. \tag{49}$$

And the light-cone wave function for the light mesons is defined as

$$\langle M_2|\overline{\chi}\overline{W}\not{n}\gamma_5\delta(\eta-\mathcal{Q}_+)\overline{W}^\dagger\chi\rangle|0\rangle$$

$$= -if_{M2}2E\int_0^1 du\delta[\eta-(4u-2)E]\phi_{M2}(u). \tag{50}$$

For $\gamma_\nu(1-\gamma_5)\otimes\gamma^\mu(1\mp\gamma_5)$, the matrix element which is given by

$$\begin{aligned}\langle T_i^{(1)}\rangle =& \frac{iC_F\pi\alpha_s}{N^2}f_Bf_{M1}f_{M2}m_B \\ &\times\int dr_+\frac{\phi_B^+(r_+)}{r_+}\int du\frac{\phi_{M_1}(u)}{u}\int dv\frac{\phi_{M2}(v)}{v}.\end{aligned} \tag{51}$$

For $(1-\gamma_5)\otimes(1+\gamma_5)$, the matrix element is zero if we use the leading-twist B meson wave function because

$$\begin{aligned}\langle 0|\overline{q}_s\not{n}\gamma_\perp^\mu(1+\gamma_5)h|B\rangle =& -\frac{if_Bm_B}{4}\mathrm{tr}\ \Big[\not{n}\gamma_\perp^\mu(1+\gamma_5)\frac{1+\not{v}}{2}\Big(\not{\overline{n}}\phi_B^+ + \not{n}\phi_B^-\Big)\gamma_5\Big] \\ =& -\frac{if_Bm_B}{8}\phi_B^+\mathrm{tr}\ \not{v}\not{\overline{n}}\not{n}\gamma_\perp^\mu = 0.\end{aligned} \tag{52}$$

If we use higher-twist wave function for the B meson, there may be nonzero contributions, but this is expected to be suppressed.

6 Spectator contribution to the form factor

In Section 5, we have considered the nonfactorizable spectator contributions arising from the subleading operators in which we include only the subleading part from the light-to-light current. However, there is also a subleading operator coming from the heavy-to-light current, but this contributes to the heavy-to-light form factor. It is considered first in Ref. [20], and we discuss in detail here in the context of nonleptonic B decays.

The operators in $\mathrm{SCET_I}$, which contribute to the form factor, are given by

$$\begin{aligned}J_i^{(0)} &= (\overline{\xi}W\Gamma_{1i}h)(\overline{\chi}\overline{W}\Gamma_{2i}\overline{W}^\dagger\chi), \\ J_i^{(1a)} &= \Big(\overline{\xi}W(W^\dagger i\overleftarrow{\not{\mathcal{D}}}_{n\perp}W)\frac{\Gamma_{1i}}{\overline{n}\cdot\mathcal{P}^\dagger}h\Big)(\overline{\chi}\overline{W}\Gamma_{2i}\overline{W}^\dagger\chi), \\ J_i^{(1b)} &= \Big(\overline{\xi}W(W^\dagger i\overrightarrow{\not{\mathcal{D}}}_{n\perp}W)\frac{\Gamma_{1i}}{m_b}h\Big)(\overline{\chi}\overline{W}\Gamma_{2i}\overline{W}^\dagger\chi).\end{aligned} \tag{53}$$

Here we list only singlet operators. The nonsinglet operators give the same matrix elements as the singlet operators with the color suppression factor $1/N$. We will consider only singlet operators from now on.

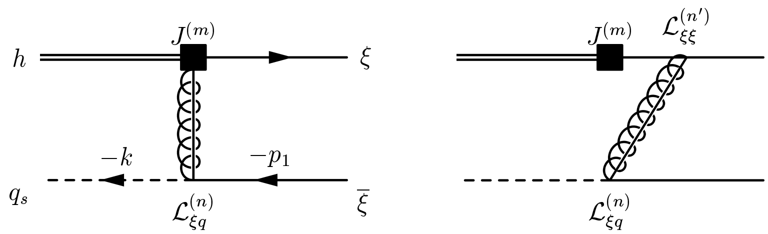

Fig. 5. Tree-level graphs in $\mathrm{SCET_I}$ for the spectator contribution to the heavy-to-light form factor. The first diagram contributes to $T_{1,2,4}$, and the second diagram contributes to $T_{0,1,3,4,5,6}$.

The interaction of collinear and usoft quarks is given by [20]

$$\begin{aligned}
\mathcal{L}^{(1)}_{\xi q} &= ig\bar{\xi}\frac{1}{i\bar{n}\cdot D_n}\slashed{\mathcal{B}}^n_\perp W q_{us} + \text{h.c.},\ \mathcal{L}^{(2a)}_{\xi q} = ig\bar{\xi}\frac{1}{i\bar{n}\cdot D_n}\slashed{M} W q_{us} + \text{h.c.},\\
\mathcal{L}^{(2b)}_{\xi q} &= ig\bar{\xi}\frac{\bar{\slashed{n}}}{2} i\slashed{D}^n_\perp \frac{1}{(i\bar{n}\cdot D_n)^2}\slashed{\mathcal{B}}^n_\perp W q_{us} + \text{h.c.},
\end{aligned} \tag{54}$$

where

$$ig\slashed{\mathcal{B}}^n_\perp = [i\bar{n}\cdot D_n, i\slashed{D}^n_\perp],\ ig\slashed{M} = [i\bar{n}\cdot D^n, i\slashed{D}^{us} + \frac{\bar{\slashed{n}}}{2} g n\cdot A_n]. \tag{55}$$

At leading order in SCET, the relevant time-ordered products are given as

$$\begin{aligned}
T^F_{0i} &= \int d^4x T[J^{(0)}_i(0) i\mathcal{L}^{(1)}_{\xi q}(x)],\ T^F_{1i} = \int d^4x T[J^{(1a)}_i(0) i\mathcal{L}^{(1)}_{\xi q}(x)],\\
T^F_{2i} &= \int d^4x T[J^{(1b)}_i(0) i\mathcal{L}^{(1)}_{\xi q}(x)],\ T^F_{3i} = \int d^4x T[J^{(0)}_i(0) i\mathcal{L}^{(2b)}_{\xi q}(x)],\\
T^{NF}_{4i} &= \int d^4x T[J^{(0)}_i(0) i\mathcal{L}^{(2a)}_{\xi q}(x)],\\
T^{NF}_{5i} &= \int d^4x d^4y T[J^{(0)}_i(0) i\mathcal{L}^{(1)}_{\xi\xi}(x) i\mathcal{L}^{(1)}_{\xi q}(y)],\\
T^{NF}_{6i} &= \int d^4x d^4y T[J^{(0)}_i(0) i\mathcal{L}^{(1)}_{cg}(x) i\mathcal{L}^{(1)}_{\xi q}(y)],
\end{aligned} \tag{56}$$

where $\mathcal{L}^{(1)}_{\xi\xi}$ is the leading collinear Lagrangian and $\mathcal{L}^{(1)}_{cg}$ is the subleading gluon Lagrangian, which can be found in Refs. [20,21]. The Feynman diagrams at lowest order in α_s for the time-ordered products are shown in Fig. 5 schematically. Compared to the case of the nonfactorizable spectator interactions, the leading-order heavy-to-light current $J^{(0)}_i$ contains A^μ_n, therefore it contributes to the heavy-to-light form factor starting from the leading order.

As suggested in Ref. [20], we absorb the nonfactorizable parts T^{NF}_{ki} $(k = 4, 5, 6)$ into the definition of the soft nonperturbative effects for the form factors at this order. Among the factorizable contributions T^F_{li}, only T^F_{2i} is nonzero at order α_s. For $\Gamma_{1i}\otimes\Gamma_{2i}=\gamma_\nu(1-\gamma_5)\otimes\gamma^\nu(1-\gamma_5)$, the matrix element of T_{2i}^F is given by

$$\begin{aligned}
\langle T_{2i}^F\rangle &= \langle M_1|i\int d^4x T[J_i^{(1b)}(0),\mathcal{L}_{\xi q}^{(1)}(x)]|B\rangle \\
&= \frac{g^2}{4\pi}\frac{1}{m_b\overline{n}\cdot p_1}\int d\overline{n}\cdot x\int\frac{dn\cdot k}{n\cdot k}e^{in\cdot k\overline{n}\cdot x/2} \\
&\times\frac{C_F}{4N}\langle\overline{\xi}W\overline{\not n}(1-\gamma_5)W^\dagger\xi\cdot\overline{\chi}\overline{W}\not n(1-\gamma_5)\overline{W}^\dagger\chi\cdot\overline{q}_sS(\overline{n}\cdot x)\not n(1-\gamma_5)S^\dagger h\rangle \\
&= \frac{\alpha_s}{4\pi}\frac{4\pi^2C_F}{N}if_{M_1}f_{M2}f_B\frac{2E}{m_b}m_B\int du\frac{\phi_{M_1}(u)}{u}\int\frac{dr_+}{r_+}\phi_B^+(r_+),
\end{aligned}\tag{57}$$

where we use Fierz transformations successively. For $\Gamma_{1i}\otimes\Gamma_{2i}=\gamma_\nu(1-\gamma_5)\otimes\gamma^\nu(1+\gamma_5)$, we obtain the same result as in Eq. (57). For $(1+\gamma_5)\otimes(1-\gamma_5)$, it vanishes.

The form factors for B decays into light pseudoscalar mesons are defined as

$$\begin{aligned}
\langle P(p)|\overline{q}\gamma^\mu b|\overline{B}(p_B)\rangle &= f_+(q^2)\Big[p_B^\mu+p^\mu-\frac{m_B^2-m_P^2}{q^2}q^\mu\Big] \\
&\quad +f_0(q^2)\frac{m_B^2-m_P^2}{q^2}q^\mu,
\end{aligned}\tag{58}$$

where $q^\mu=p_B^\mu-p^\mu$. In SCET, the form factor at order α_s is given by

$$\begin{aligned}
f_+(0) &= \pi\alpha_s\frac{C_F}{N}\frac{f_{M1}f_Bm_B}{4E^2}\frac{2E}{m_b}\int dxdr_+\frac{\alpha_s(\mu)}{xr_+}\phi_{M1}(x,\mu)\phi_B^+(r_+,\mu) \\
&\quad +(1+K)\zeta(\mu_0,\mu),
\end{aligned}\tag{59}$$

where $\zeta(\mu_0,\mu)$ is a nonperturbative function introduced in Ref. [24] in large-energy effective theory [25]. A similar nonperturbative function is introduced in Refs. [9,13], based on SCET. The procedure in obtaining independent nonperturabtive functions in the form factor is different in the large-energy effective theory and in SCET, but the number of independent nonperturbative functions is the same. And K at $\mu=m_b$ is given by

$$K=-\frac{\alpha_sC_F}{4\pi}\Big(6+\frac{\pi^2}{12}\Big).\tag{60}$$

The expression for f_+ coincides with the result in Ref. [20] with $m_B=2E$ at order α_s, and K is calculated in Refs. [9,13].

7 Application to $\overline{B} \to \pi\pi$ decays

As an application, we consider the decay amplitudes for $\overline{B} \to \pi\pi$, which can be written as

$$\langle \pi\pi | H_{\text{eff}} | \overline{B} \rangle = \frac{G_F}{\sqrt{2}} \sum_{p=u,c} V_{pb} V_{pd}^* \langle \pi\pi | \mathcal{A}_p | \overline{B} \rangle, \tag{61}$$

where the operators $\mathcal{A}_p$ are given by

$$\begin{aligned}
\mathcal{A}_p = & a_1^p \Big[(\overline{\xi}^u h)_{V-A} (\overline{\chi}^d \chi^u)_{V-A} \Big] + a_2^p \Big[(\overline{\xi}^d h)_{V-A} (\overline{\chi}^u \chi^u)_{V-A} \Big] \\
& + a_3^p \Big[(\overline{\xi}^d h)_{V-A} (\overline{\chi}^q \chi^q)_{V-A} \Big] + a_4^p \Big[(\overline{\xi}^q h)_{V-A} (\overline{\chi}^d \chi^q)_{V-A} \Big] \\
& + a_5^p \Big[(\overline{\xi}^d h)_{V-A} (\overline{\chi}^q \chi^q)_{V+A} \Big].
\end{aligned} \tag{62}$$

The notation $a_i^p[\mathcal{O}]$ means that a_i^p are the sum of the amplitudes initiated by the operator $\mathcal{O}$. They include the contribution from the operator itself, and the spectator contributions. That is, $a_i^p[\mathcal{O}]$ can be written as

$$a_i^p[\mathcal{O}] = T_i^p + N_i^p + F_i^p, \tag{63}$$

where T_i^p is the contribution from the four-quark operators, N_i^p is the nonfactorizable spectator contribution, and F_i^p is the spectator contribution for the heavy-to-light form factor with $a_1^c = a_2^c = 0$.

Before we present the decay amplitudes for $\overline{B} \to \pi\pi$ at order α_s explicitly, we show the amplitudes a_i^p in general to all orders in α_s, which are derived from SCET. The form of T_i^p from Eq. (30) can be written as

$$\begin{aligned}
T_i^p = & \int d\eta \; C_{\text{eff},i}^T(\eta, \mu_0, \mu) \\
& \times \langle \overline{\xi} W \gamma_\mu (1-\gamma_5) S^\dagger h \cdot \overline{\chi} \overline{W} \delta(\eta - \mathcal{Q}_+) \gamma^\mu (1 \mp \gamma_5) \overline{W}^\dagger \chi \rangle \\
= & \pm i f_{M2} 2E \int_0^1 du C_{\text{eff},i}^T(u, \mu_0, \mu) \phi_{M2}(u, \mu) \langle M_1 | \overline{\xi} W \frac{\not{\overline{n}}}{2} (1-\gamma_5) S^\dagger h | \overline{B} \rangle \\
= & \pm i m_B^2 f_{M2} \int_0^1 du \zeta(\mu_0, \mu) C_{\text{eff},i}^T(u, \mu_0, \mu) \phi_{M2}(u, \mu),
\end{aligned} \tag{64}$$

where the upper (lower) sign corresponds to $i = 1, \cdots, 4$ $(i = 5)$. The effective Wilson coefficients $C_{\text{eff},i}^T$ are evaluated at $\mu = m_b$ and evolved down to $\mu = \mu_0 = \sqrt{m_b \Lambda}$ to be matched onto SCET_{II}. Then they evolve down to the scale μ, where the matrix elements are evaluated.

The matrix element of the heavy-to-light current is given by [24,13,9]

$$\langle M_1|\overline{\xi}W\overline{\not{n}}(1-\gamma_5)S^\dagger h|\overline{B}\rangle = 2m_B\zeta. \tag{65}$$

The nonperturbative parameter ζ is matched at $\mu=\mu_0$ and it evolves down to μ, and the wave functions are evaluated at μ. Note that we do not include the radiative corrections in ζ, since they are taken into account either in the Wilson coefficients or in the spectator contributions.

The spectator contribution N_i^p involves the time-ordered products, which can be written as

$$\begin{aligned}T[O_i^{(1a)}(x)+O_i^{(1b)}(x), i\mathcal{L}_{\xi q}^{(1)}(0)] = \delta\Big(\frac{n\cdot x}{2}\Big)\delta^2(x_\perp)\int d\overline{\eta}d\eta\int dr_+ e^{ir_+\overline{n}\cdot x/2}\\ \times J_i^N(\eta,\overline{\eta},r_+)\mathcal{O}_i(\eta,\overline{\eta},r_+),\end{aligned} \tag{66}$$

where J_i^N are the jet functions, which are obtained by matching $\mathrm{SCET_I}$ onto $\mathrm{SCET_{II}}$. The index i denotes all the possible operators for the decay. And the operator $\mathcal{O}_i$ is given by

$$\begin{aligned}\mathcal{O}_i(\eta,\overline{\eta},r_+) = &\overline{\xi}W\Gamma_{1i}\delta(\eta-\mathcal{P}_+)W^\dagger\xi\cdot\overline{\chi}\overline{W}\Gamma_{2i}\delta(\overline{\eta}-\mathcal{Q}_+)\overline{W}^\dagger\chi\\ &\times\overline{q}_s\not{n}(1-\gamma_5)\delta(\mathcal{R}^\dagger-r_+)S^\dagger h.\end{aligned} \tag{67}$$

Therefore N_i^p can be written as

$$\begin{aligned}N_i^p &= \int d^4x C_{\mathrm{eff},i}^N T[O_i^{(1a)}(x)+O_i^{(1b)}(x), i\mathcal{L}_{\xi q}^{(1)}(0)]\\ &= \int dudvdr_+ C_{\mathrm{eff},i}^N(\mu_0,\mu)J_i^N(u,v,r_+,\mu_0,\mu)N_i f_B f_{M1} f_{M_2}\\ &\quad\times\phi_{M1}(u,\mu)\phi_{M2}(v,\mu)\phi_B^+(r_+,\mu),\end{aligned} \tag{68}$$

where $C_{\mathrm{eff},i}^N$ are the effective Wilson coefficients, and the μ dependence is shown explicitly. Here N_i are the normalization constants when we evaluate the matrix elements of $\mathcal{O}_i$. Since $\overline{n}\cdot p_{M1} = n\cdot p_{M2} = 2E$, the variables u and v satisfy the relations

$$u = \frac{\eta}{4E}+\frac{1}{2},\quad v = \frac{\overline{\eta}}{4E}+\frac{1}{2}. \tag{69}$$

The factorizable spectator contribution to the form factor can be written similarly as

$$F_i^p = \int dudvdr_+ C_{\mathrm{eff},i}^F(\mu_0,\mu)J_i^F(u,v,r_+,\mu_0,\mu)N_i f_B f_{M1} f_{M2}$$

$$\times \phi_{M1}(u,\mu)\phi_{M2}(v,\mu)\phi_B^+(r_+,\mu), \tag{70}$$

where $C^F_{\mathrm{eff},i}$ are the effective Wilson coefficients, and J_i^F are the jet functions obtained from matching $\mathrm{SCET_I}$ onto $\mathrm{SCET_{II}}$. The spectator interactions N_i^p in Eq. (68) and F_i^p in Eq. (70) are factorized into the short-distance and the long-distance parts to all orders in α_s, and the convolution integrals are finite.

Now let us calculate the decay amplitudes for $\overline{B} \to \pi\pi$ at order α_s, based on the general expressions on T_i^p, N_i^p and F_i^p. The contributions T_i^p are obtained by the convolutions of the following effective Wilson coefficients

$$\begin{aligned} C^T_{\mathrm{eff},1} &= C_{1R} + \frac{C_{2R}}{N},\ C^T_{\mathrm{eff},2} = C_{2C} + \frac{C_{1C}}{N},\ C^T_{\mathrm{eff},3} = C_{3R} + \frac{C_{4R}}{N}, \\ C^{Tp}_{\mathrm{eff},4} &= C^p_{4C} + \frac{C^p_{3C}}{N},\ C^{Tp}_{\mathrm{eff},5} = C_{5R} + \frac{C_{6R}}{N}, \end{aligned} \tag{71}$$

with the hadronic matrix elements of the four-quark operators. For the decay amplitudes at leading order, we can put $x=1$ and let $u = x_1$ and $\overline{u} = x_2 = 1-u$, and the amplitudes T_i^p's, evaluated at $\mu = m_b$ are given as

$$\begin{aligned} T_1^p &= i m_B^2 \zeta f_\pi \Big[\Big(C_1 + \frac{C_2}{N}\Big)(1+K) + \frac{\alpha_s}{4\pi}\frac{C_F}{N} C_2 F\Big], \\ T_2^p &= i m_B^2 \zeta f_\pi \Big[\Big(C_2 + \frac{C_1}{N}\Big)(1+K) + \frac{\alpha_s}{4\pi}\frac{C_F}{N} C_1 F\Big], \\ T_3^p &= i m_B^2 \zeta f_\pi \Big[\Big(C_3 + \frac{C_4}{N}\Big)(1+K) + \frac{\alpha_s}{4\pi}\frac{C_F}{N} C_4 F\Big], \\ T_4^p &= i m_B^2 \zeta f_\pi \Big\{\Big[\Big(C_4 + \frac{C_3}{N}\Big)(1+K) + \frac{\alpha_s}{4\pi}\frac{C_F}{N}\Big[C_3 F + C_1\Big(\frac{2}{3} - G(s_p)\Big) \\ &\quad + C_3\Big(\frac{4}{3} - G(0) - G(1)\Big) + C_4\Big(\frac{2n_f}{3} - 3G(0) - G(s_c) - G(1)\Big) \\ &\quad + C_6\Big(-3G(0) - G(s_c) - G(1)\Big) + G_{\pi,8}(C_5 + C_8)\Big]\Big\}, \\ T_5^p &= -i m_B^2 \zeta f_\pi \Big[\Big(C_5 + \frac{C_6}{N}\Big)(1+K) + \frac{\alpha_s}{4\pi}\frac{C_F}{N} C_6(-F - 12)\Big], \end{aligned} \tag{72}$$

where $C_F = (N^2-1)/(2N)$, $N=3$, $n_f = 5$. And F is given as

$$\begin{aligned} F &= -18 + f_\pi^{\mathrm{I}}, \quad f_\pi^{\mathrm{I}} = \int_0^1 du g(u)\phi_\pi(u), \quad G_{\pi,8} = \int_0^1 du \frac{-2}{1-u}\phi_\pi(u), \\ g(u) &= 3\frac{1-2u}{1-u}\ln u - 3i\pi \\ &\quad - \Big[2\mathrm{Li}_2(1-u) + \frac{1-3u}{1-u}\ln u + \ln^2 u - (u \leftrightarrow \overline{u})\Big]. \end{aligned} \tag{73}$$

The nonfactorizable spectator contributions N_i^p are given by Eq. (51) as

$$N_i^p = i\frac{C_F}{N^2}\pi\alpha_s f_B f_\pi^2 m_B C_{\mathrm{eff},i}^N \int dr_+ \frac{\phi_B^+(r_+)}{r_+}\Big(\int_0^1 \frac{\phi_\pi(u)}{u}\Big)^2, \tag{74}$$

where the effective Wilson coefficients are given by

$$C_{\mathrm{eff},1}^N = C_2,\ C_{\mathrm{eff},2}^N = C_1,\ C_{\mathrm{eff},3}^N = C_4,\ C_{\mathrm{eff},4}^N = C_3,\ C_{\mathrm{eff},5}^N = C_6. \tag{75}$$

The spectator contributions to the heavy-to-light form factor F_i^p are given as

$$F_i^p = i\frac{C_F}{N}\pi\alpha_s f_B f_\pi^2 m_B C_{\mathrm{eff},i}^F \int du \frac{\phi_\pi(u)}{u}\int dr_+ \frac{\phi_B^+(r_+)}{r_+}, \tag{76}$$

where the effective Wilson coefficients are given by

$$\begin{aligned} C_{\mathrm{eff},1}^F &= C_1 + \frac{C2}{N},\ C_{\mathrm{eff},2}^F = C_2 + \frac{C_1}{N},\ C_{\mathrm{eff},3}^F = C_3 + \frac{C_4}{N}, \\ C_{\mathrm{eff},4}^F &= C_4 + \frac{C_5}{N},\ C_{\mathrm{eff},5}^F = C_5 + \frac{C_6}{N}. \end{aligned} \tag{77}$$

The final expression can be simplied when we use the definition of f_+ given in Eq. (59). For example, $T_1^p + F_1^p$ is written as

$$\begin{aligned} T_1^p + F_1^p &= im_B^2\zeta f_\pi\Big[\Big(C_1 + \frac{C_2}{N}\Big)(1+K) + \frac{\alpha_s}{4\pi}\frac{C_F}{N}C_2F\Big], \\ &\qquad + i\frac{C_F}{N}\pi\alpha_s f_\pi^2 f_B m_B\Big(C_1 + \frac{C_2}{N}\Big)\int du\frac{\phi_\pi(u)}{u}\int dr_+\frac{\phi_B^+(r_+)}{r_+} \\ &= im_B^2\zeta f_\pi\frac{\alpha_s}{4\pi}\frac{C_F}{N}C_2F + im_B^2 f_\pi\Big(C_1 + \frac{C_2}{N}\Big)\Big[\zeta(1+K) \\ &\qquad + \pi\alpha_s\frac{C_F}{N}\frac{f_\pi f_B}{m_B}\int du\frac{\phi_\pi(u)}{u}\int dr_+\frac{\phi_B^+(r_+)}{r_+}\Big] \\ &\approx im_B^2 f_+ f_\pi\Big[\Big(C_1 + \frac{C_2}{N}\Big) + \frac{\alpha_s}{4\pi}\frac{C_F}{N}C_2F\Big], \end{aligned} \tag{78}$$

where the definition of f_+ in Eq. (59) is used in the last line with $m_b \approx m_B = 2E$. And we replace ζ by f_+ in the term proportional to F. This induces terms of $\mathcal{O}(\alpha_s^2)$, which is neglected. This relation also holds for the combinations $T_i^p + F_i^p$ for all i.

In summary, the decay amplitudes a_i^p are given by

$$
\begin{aligned}
a_1^p =& im_B^2 f_+ f_\pi \Big[\Big(C_1+\frac{C_2}{N}\Big)+\frac{\alpha_s}{4\pi}\frac{C_F}{N}C_2F'\Big],\\
a_2^p =& im_B^2 f_+ f_\pi \Big[\Big(C_2+\frac{C_1}{N}\Big)+\frac{\alpha_s}{4\pi}\frac{C_F}{N}C_1F'\Big],\\
a_3^p =& im_B^2 f_+ f_\pi \Big[\Big(C_3+\frac{C_4}{N}\Big)+\frac{\alpha_s}{4\pi}\frac{C_F}{N}C_4F'\Big],\\
a_4^p =& im_B^2 f_+ f_\pi \Big\{\Big[\Big(C_4+\frac{C_3}{N}\Big)+\frac{\alpha_s}{4\pi}\frac{C_F}{N}\Big[C_3F'+C_1\Big(\frac{2}{3}-G(s_p)\Big)\\
&+C_3\Big(\frac{4}{3}-G(0)-G(1)\Big)+C_4\Big(\frac{2n_f}{3}-3G(0)-G(s_c)-G(1)\Big)\\
&+C_6\Big(-3G(0)-G(s_c)-G(1)\Big)+G_{\pi,8}(C_5+C_8)\Big]\Big\},\\
a_5^p =& -im_B^2 f_+ f_\pi \Big[\Big(C_5+\frac{C_6}{N}\Big)+\frac{\alpha_s}{4\pi}\frac{C_F}{N}C_6(-F'-12)\Big],
\end{aligned}
\tag{79}
$$

where

$$
F' = F + \frac{4\pi^2}{N}\frac{f_\pi f_B}{f_+ m_B^2} m_B \int_0^1 dr_+ \frac{\phi_B^+(r_+)}{r_+}\Big(\int_0^1 \frac{\phi_\pi(u)}{u}\Big)^2. \tag{80}
$$

The decay amplitudes in Eq. (79) at order α_s are consistent with the result obtained by Beneke et al. [7] at order α_s. However, our result goes further in the sense that we consider the operators to all orders in α_s, as shown in Eqs. (64), (68) and (70). At higer orders, our result will be different from the result in Ref. [7] since there are two scales m_b and $\mu_0 = \sqrt{m_b\Lambda}$ involved and the effects of these two scales cannot be reproduced in the heavy quark limit.

8 Conclusion

We have considered the four-quark operators relevant to nonleptonic B decays into two light mesons in SCET at leading order in Λ and to next-to-leading order in α_s. The construction of the four-quark operators in SCET is process-dependent since we first have to specify the directions of the outgoing quarks and construct the operators accordingly. In matching onto $\mathrm{SCET_{II}}$, we integrate out off-shell modes by attaching collinear and soft gluons to fermion lines. The result is given as gauge-invariant four-quark operators. The form of the gauge-invariant operators is obtained to all orders in α_s by using the auxiliary field method. The Wilson coefficients of these operators can be computed by matching the amplitudes between the full theory and $\mathrm{SCET_I}$ since the infrared divergence of the full theory is reproduced in $\mathrm{SCET_I}$. And we obtain jet functions through the matching between $\mathrm{SCET_I}$ and $\mathrm{SCET_{II}}$.

When the effects of collinear and soft gluons are included, we can obtain gauge-invariant operators, and the explicit form of these operators guarantees the color transparency at leading order in Λ but to all orders in α_s. Now the idea of the naive factorization in which the matrix elements of four-quark operators are reduced to products of the matrix elements of two currents has a theoretical basis. Furthermore when we include spectator interactions which contribute to the nonfactorizable contribution and to the heavy-to-light form factor, the amplitudes, which include four-quark and six-quark operators, are factorized to all orders in α_s. That is, the amplitudes can be written as a convolution of short-distance effects represented by the effective Wilson coefficients and long-distance effects represented by the light-cone wave functions of mesons. And the convolution integrals appearing in the hard spectator interactions and in the hard contribution to the form factors are finite. Therefore we have proved that the decay amplitudes for nonleptonic B decays into two light mesons at leading order in SCET and to all orders in α_s are factorized.

Note that we have not included renormalization group running of the effective Wilson coefficients. In order to include the renormalization group running, the scaling of the Wilson coefficients can be achieved by calculating the anomalous dimensions of the operators, say, from Eqs. (25) and (27). But the running of the Wilson coefficients may not be appreciable at this order for the scale change from $\mu = m_b$ to $\mu = \sqrt{m_b \Lambda}$.

As an application of SCET, we have calculated the decay amplitudes for $\overline{B} \to \pi\pi$ at leading order in SCET. The results are consistent at order α_s with the result in the heavy quark mass limit, which is shown explicitly here. This is not a coincidence because the leading-order decay amplitudes in the heavy quark mass limit employing leading-twist meson wave functions correspond to the leading-order decay amplitudes in SCET. However SCET extends the analysis to all orders in α_s, and proves that the decay amplitudes are factorized. The two types of the factorization properties, which correspond to the color transparency and the separation of long- and short-distance effects are satisfied to all orders in α_s in nonleptonic B decays.

We can go beyond the leading-order calculation and consider subleading corrections in order to check the validity of the approach using SCET. For example, we can ask questions on how chirally-enhaced contributions can be treated in SCET, or how to include higher-twist wave functions of mesons, and how we can organize higher-order corrections in SCET. However we stress that this is a first step toward understanding nonleptonic B decays, and those questions are under investigation.

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A Derivation of the gauge-invariant four-quark operators using the auxiliary field method

In Section 2 we derived gauge-invariant four-quark operators through the matching calculation to order g^2, in which all the off-shell modes were integrated out. We can derive the gauge-invariant four-quark operators of the form $(\overline{\xi}\Gamma_1 h)\cdot(\overline{\chi}\Gamma_2\chi)$ using the auxiliary field method. We construct a Lagrangian of the on-shell fields with the off-shell fields which are auxiliary fields. Then we integrate out the off-shell auxiliary fields to obtain gauge-invariant operators entirely in terms of the on-shell fields. In integrating out the off-shell modes, the Wilson lines W, $\overline{W}$, S and $\overline{S}$ appear and the final form is gauge invariant. The auxiliary field method offers a consistent way to derive gauge-invariant operators to all orders.

The auxiliary field method has been used in SCET for the gauge-invariant heavy-to-light current operators [15], and for the processes in which there are collinear particles in the n^μ and $\overline{n}^\mu$ directions [12]. We extend the idea of the auxiliary field method to derive the gauge-invariant four-quark operators in nonleptonic decays. Since there are collinear fields both in the n^μ and the $\overline{n}^\mu$ directions, the auxiliary field method for collinear quarks ξ and χ is the same as the method presented in Ref. [12]. Here we consider how to treat the heavy quark interacting with two types of collinear gluons A_n^μ and $A_{\overline{n}}^\mu$.

For clarity, we first construct the Lagrangian of the on-shell fields with the off-shell fields of order $p^2 \sim Q^2$, and integrate out the off-shell modes to obtain the gauge-invariant four-quark operators. Through this procedure, we obtain the Lagrangian and the four-quark operators in $\mathrm{SCET_I}$. Once we get familiar with the construction of the Lagrangian with the off-shell modes of order $p^2 \sim Q^2$, we include the off-shell modes of order $p^2 \sim Q\Lambda$, and integrate out all the off-shell modes. We can obtain the Lagrangian and the four-quark operators in $\mathrm{SCET_{II}}$ in a gauge-invariant way.

Let us categorize all the fields in $\mathrm{SCET_I}$ in powers of Λ. Note that the small expansion parameter λ in $\mathrm{SCET_I}$ is of order $\sqrt{\Lambda/Q}$ and λ' in $\mathrm{SCET_{II}}$ is of order Λ/Q. In order to avoid confusion, we express the scaling of all the momenta in powers of Λ. The on-shell fields are the collinear fields ξ, A_n^μ, which scale as $p_n^\mu = (n\cdot p, \overline{n}\cdot p, p_\perp) \sim (\Lambda, Q, \sqrt{Q\Lambda})$, the collinear fields χ, $A_{\overline{n}}^\mu$, which scale as

Table A.1
List of fields to construct the auxiliary Lagrangian in $\mathrm{SCET_I}$. The on-shell fields are collinear, and heavy quark fields, and the Wilson lines obtained by the corresponding gluons are listed in the last column. The off-shell fields, which appear as intermediate states, are classified by their momentum scaling behavior.

	Momentum scaling	Fields	Wilson lines
on-shell	$p_n^\mu \sim (\Lambda, Q, (Q\Lambda)^{1/2})$	ξ, A_n^μ	W
	$p_{\overline{n}}^\mu \sim (Q, \Lambda, (Q\Lambda)^{1/2})$	χ, $A_{\overline{n}}^\mu$	$\overline{W}$
	$p_{us}^\mu \sim (\Lambda, \Lambda, \Lambda)$	h	
off-shell	$p_n^\mu + p_{\overline{n}}^\mu \sim (Q, Q, (Q\Lambda)^{1/2})$	ψ_H, A_Q^μ, ξ_Q, χ_Q	W_Q, $\overline{W}_Q$
	$p_n^\mu \sim (\Lambda, Q, (Q\Lambda)^{1/2})$	ψ_n	
	$p_{\overline{n}}^\mu \sim (Q, \Lambda, (Q\Lambda)^{1/2})$	$\psi_{\overline{n}}$	

$(Q, \Lambda, \sqrt{Q\Lambda})$, the soft fields q_s, A_s^μ, which scale as $p_s^\mu \sim \sqrt{Q\Lambda}$, and the usoft fields q_{us}, A_{us}^μ, h, which scale as $p_{us}^\mu \sim \Lambda$.

In $\mathrm{SCET_I}$, we include the following off-shell fields. When the on-shell heavy quark h interacts with a collinear gluon A_n^μ ($A_{\overline{n}}^\mu$), the final heavy quark has momentum of order $p_n^\mu \sim (\Lambda, Q, \sqrt{Q\Lambda})$ [$p_{\overline{n}}^\mu \sim (Q, \Lambda, \sqrt{Q\Lambda})$]. We label this off-shell heavy quark as ψ_n ($\psi_{\overline{n}}$). If the on-shell heavy quark h interacts with A_n^μ and $A_{\overline{n}}^\mu$, the final off-shell heavy quark has momentum of order $p_n^\mu + p_{\overline{n}}^\mu \sim (Q, Q, \sqrt{Q\Lambda})$, and $p^2 \sim Q^2$. We label this off-shell field as ψ_H. A gluon interacting with A_n^μ and $A_{\overline{n}}^\mu$ through the triple gluon vertex also has a momentum $p_n^\mu + p_{\overline{n}}^\mu$, and we label this gluon as A_Q^μ. We list all the on-shell fields and the off-shell auxiliary fields in Table A.1. The collinear quarks can also be off-the mass shell of order $p^2 \sim Q^2$, which we denote as ξ_Q and χ_Q. The treatment of the off-shell collinear quarks is presented in Ref. [12], and we will not repeat it here. Since the soft and usoft quarks and gluons are not relevant in our argument, we do not present them in Table A.1.

The auxiliary gluon Lagrangian in $\mathrm{SCET_I}$ is given by

$$\mathcal{L}_{\rm aux}^g[A_Q] = \frac{1}{2g^2}\mathrm{tr}\Big([iD_Q^\mu + gA_Q^\mu, iD_Q^\nu + gA_Q^\nu]\Big)^2 + \frac{1}{\alpha_L}\mathrm{tr}\Big(\Big[iD_{Q\mu}, A_Q^\mu]\Big)^2, \quad \text{(A.1)}$$

where the covariant derivative D_Q^μ is given by

$$\begin{aligned} iD_Q^\mu = & \frac{n^\mu}{2}(\overline{n}\cdot\mathcal{P} + g\overline{n}\cdot A_n) + \frac{\overline{n}^\mu}{2}(n\cdot\mathcal{Q} + gn\cdot A_{\overline{n}}) \\ & + \mathcal{P}_\perp^\mu + gA_{n\perp}^\mu + \mathcal{Q}_\perp^\mu + gA_{\overline{n}\perp}^\mu, \end{aligned} \quad \text{(A.2)}$$

and only the leading terms in Eq. (A.2) is included in $\mathcal{L}_{\rm aux}^g$ at leading order.

We can separately obtain the solution of the first term and the second term in Eq. (A.1). The equation of motion of the first term is given by

$$[iD_{Q\mu} + gA_{Q\mu}, [iD_Q^\mu + gA_Q^\mu, iD_Q^\nu + gA_Q^\nu]] = 0. \tag{A.3}$$

The leading-order solution is obtained by making an ansatz

$$\overline{W}_Q^\dagger W_Q = W\overline{W}^\dagger. \tag{A.4}$$

Here W_Q and $\overline{W}_Q$ are essentially the Fourier transform of the Wilson lines

$$\begin{aligned} W_Q(y) &= \mathrm{P}\exp\Big\{ig\int_{-\infty}^{y} ds\Big[\overline{n}\cdot A_Q(s\overline{n}) + \overline{n}\cdot A_n(s\overline{n})\Big]\Big\}, \\ \overline{W}_Q(y) &= \mathrm{P}\exp\Big\{ig\int_{-\infty}^{y} ds\Big[n\cdot A_Q(sn) + n\cdot A_{\overline{n}}(sn)\Big]\Big\}, \end{aligned} \tag{A.5}$$

which satisfy

$$\overline{n}\cdot\Big(\mathcal{P} + gA_Q + gA_n\Big)W_Q = 0,\ n\cdot\Big(\mathcal{Q} + gA_Q + gA_{\overline{n}}\Big)\overline{W}_Q = 0. \tag{A.6}$$

The Lagrangian with auxiliary heavy fields is complicated because only the sum of three graphs shown in Eq. (16) is simple, but the individual diagrams show complex behavior. Recall that Eq. (16) is given by

$$M_a + M_b = \frac{ig^2}{2} f_{abc}\overline{q}\Gamma_1 T_a \frac{n\cdot A_{\overline{n}}^b \overline{n}\cdot A_n^c}{n\cdot q_{\overline{n}}\overline{n}\cdot q_n} b + g^2\overline{q}\Gamma_1\frac{\overline{n}\cdot A_n}{\overline{n}\cdot q_n}\frac{n\cdot A_{\overline{n}}}{n\cdot q_{\overline{n}}} b, \tag{A.7}$$

and we construct the Lagrangian such that the first term in Eq. (A.7) is reproduced by the auxiliary field attached with the triple gluon vertex with A_n^μ and $A_{\overline{n}}^\mu$, and the second term is produced by attaching A_n^μ and $A_{\overline{n}}^\mu$ starting from the vertex in this order. It means that when a heavy quark h becomes an off-shell field ψ_H interacting with A_n^μ and $A_{\overline{n}}^\mu$, the heavy field h first interacts with $A_{\overline{n}}^\mu$ to become $\psi_{\overline{n}}$, and then interacts with A_n^μ to become ψ_H. We can choose a different convention in which h interacts with A_n^μ first and then interacts with $A_{\overline{n}}^\mu$ to become ψ_H. Both methods give the same result, as will be shown later. However, it is necessary to specify the order of the interaction since the Lagrangian for the off-shell fields is obtained by expanding the QCD Feynman rules in powers of Λ.

With the prescription described above, the Lagrangian for the heavy quark with the auxiliary fields is given as

$$
\begin{aligned}
\mathcal{L}^h_{\rm aux} = &\overline{\psi}_n g\overline{n}\cdot A_n h + \overline{\psi}_{\overline{n}} gn\cdot A_{\overline{n}} h + \overline{\psi}_H g\overline{n}\cdot A_Q (h+\psi_n) \\
&+\overline{\psi}_H g\overline{n}\cdot (A_Q + A_n)\psi_{\overline{n}} + \text{h.c.} \\
&+\overline{\psi}_n (\overline{n}\cdot\mathcal{P} + g\overline{n}\cdot A_n)\psi_n + \overline{\psi}_{\overline{n}}(n\cdot\mathcal{Q} + gn\cdot A_{\overline{n}})\psi_{\overline{n}} \\
&+\overline{\psi}_H \Big[\overline{n}\cdot\mathcal{P} + g\overline{n}\cdot(A_Q + A_n)\Big]\psi_H. \qquad \text{(A.8)}
\end{aligned}
$$

Note that there is no term in which ψ_n interacts with $A^\mu_{\overline{n}}$ to become ψ_H according to our prescription. The inclusion of the auxiliary field A^μ_Q in the second line of Eq. (A.8) should be verified at order g^3, but these terms are included since they are kinematically allowed.

Solving for $\psi_{\overline{n}}$, ψ_n and ψ_H from Eq. (A.8), we obtain

$$
\begin{aligned}
&gn\cdot A_{\overline{n}} h + n\cdot(\mathcal{Q} + gA_{\overline{n}})\psi_{\overline{n}} = 0, \qquad \text{(A.9)} \\
&g\overline{n}\cdot A_n h + \overline{n}\cdot(\mathcal{P} + gA_n)\psi_n = 0, \\
&g\overline{n}\cdot A_Q(h+\psi_n) + g\overline{n}\cdot(A_n + A_Q)\psi_{\overline{n}} + \overline{n}\cdot\Big[\mathcal{P} + g(A_n + A_Q)\Big]\psi_H = 0.
\end{aligned}
$$

The first equation in Eq. (A.9) can be solved for $\psi_{\overline{n}}$ as

$$
\psi_{\overline{n}} = (\overline{W} - 1)h, \qquad \text{(A.10)}
$$

and adding the second and the third equations in Eq. (A.9) yields

$$
\psi_n + \psi_H = (W_Q - 1)(h + \psi_{\overline{n}}). \qquad \text{(A.11)}
$$

Therefore the heavy quark field can be written as

$$
h + \psi_{\overline{n}} + \psi_n + \psi_H = h + \psi_{\overline{n}} + (W_Q - 1)(h+\psi_{\overline{n}}) = W_Q\overline{W}h. \qquad \text{(A.12)}
$$

We can arrive at an equivalent conclusion by rewriting Eq. (A.7) as

$$
M_a + M_b = \frac{ig^2}{2} f_{abc}\overline{q}\Gamma_1 T_a \frac{n\cdot A^c_{\overline{n}}\overline{n}\cdot A^b_n}{n\cdot q_{\overline{n}}\overline{n}\cdot q_n} b + g^2\overline{q}\Gamma_1 \frac{n\cdot A_{\overline{n}}}{n\cdot q_{\overline{n}}}\frac{\overline{n}\cdot A_n}{\overline{n}b\cdot q_n}, \qquad \text{(A.13)}
$$

where the order of the gluons in the second term is reversed. In this case, we require that the heavy quark h interacts with A^μ_n first and then interacts with $A^\mu_{\overline{n}}$ to generate ψ_H. The corresponding Lagrangian is given by

$$
\begin{aligned}
\mathcal{L}^h_{\rm aux} = &\overline{\psi}_n g\overline{n}\cdot A_n h + \overline{\psi}_{\overline{n}} gn\cdot A_{\overline{n}} h + \overline{\psi}_H gn\cdot A_Q (h+\psi_{\overline{n}}) \\
&+\overline{\psi}_H gn\cdot (A_Q + A_{\overline{n}})\psi_n + \text{h.c.} \\
&+\overline{\psi}_n (\overline{n}\cdot\mathcal{P} + g\overline{n}\cdot A_n)\psi_n + \overline{\psi}_{\overline{n}}(n\cdot\mathcal{Q} + gn\cdot A_{\overline{n}})\psi_{\overline{n}} \\
&+\overline{\psi}_H \Big[n\cdot\mathcal{Q} + gn\cdot(A_Q + A_{\overline{n}})\Big]\psi_H. \qquad \text{(A.14)}
\end{aligned}
$$

Solving the equations of motion, we obtain

$$h + \psi_n + \psi_{\overline{n}} + \psi_H = \overline{W}_Q W h = W_Q \overline{W} h, \tag{A.15}$$

where the last equality comes from the ansatz $\overline{W}_Q^\dagger W_Q = W\overline{W}^\dagger$. This is equivalent to the result in Eq. (A.12).

Now we construct the Lagrangian including the off-shell modes of order $p^2 \sim Q\Lambda$. Through this procedure, we obtain the Lagrangian and the four-quark operators in $\mathrm{SCET}_{\mathrm{II}}$. In $\mathrm{SCET}_{\mathrm{II}}$, we have to integrate out all the off-shell modes in Table A.2 by constructing the Lagrangian with additional auxiliary fields with additional soft momentum, with the index X. The on-shell and the off-shell fields in $\mathrm{SCET}_{\mathrm{II}}$ are listed in Table A.2. Note that the momentum scaling of each fields has changed in order to accomodate $\mathrm{SCET}_{\mathrm{II}}$. Since $p_n^\mu + p_{\overline{n}}^\mu + p_s^\mu \sim p_n^\mu + p_{\overline{n}}^\mu$, we do not put the index X to ψ_H and A_Q^μ. For the same reason, we express the off-shell heavy quarks as ψ_n and $\psi_{\overline{n}}$.

We include the off-shell modes due to the soft momentum. We label these fields with the index X to denote the off-shellness $p^2 \sim Q\Lambda$. The auxiliary Lagrangian including the off-shell modes of order $p^2 \sim Q\Lambda$ is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{aux}}^h = &\overline{\psi}_n g\overline{n}\cdot(A_n^X + A_n)h + \overline{\psi}_{\overline{n}} g n\cdot(A_{\overline{n}}^X + A_{\overline{n}})h + \overline{\psi}_H g\overline{n}\cdot A_Q(h+\psi_n)\\ &+\overline{\psi}_H g\overline{n}\cdot(A_Q + A_n^X + A_n)\psi_{\overline{n}} + \mathrm{h.c.}\\ &+\overline{\psi}_n \overline{n}\cdot\Big[\mathcal{P} + g(A_n^X + A_n)\Big]\psi_n + \overline{\psi}_{\overline{n}} n\cdot\Big[\mathcal{Q} + g(A_{\overline{n}}^X + A_{\overline{n}})\Big]\psi_{\overline{n}}\\ &+\overline{\psi}_H \overline{n}\cdot\Big[\mathcal{P} + g(A_Q + A_n)\Big]\psi_H.\end{aligned} \tag{A.16}$$

Note that there is no soft gluon A_s^μ involved in Eq. (A.16) for the heavy quark.

Table A.2
List of fields to construct the auxiliary Lagrangian in $\mathrm{SCET}_{\mathrm{II}}$. The fields with the index X respresent the off-shell fields with the soft momentum. The momentum scaling of ψ_H, A_Q^μ, ψ_n and $\psi_{\overline{n}}$ does not change with the addition of the soft momentum.

	Momentum scaling	Fields	Wilson lines
on-shell	$p_n^\mu \sim (\Lambda^2/Q, Q, \Lambda)$	ξ, A_n^μ	W
	$p_{\overline{n}}^\mu \sim (Q, \Lambda^2/Q, \Lambda)$	χ, $A_{\overline{n}}^\mu$	$\overline{W}$
	$p_s^\mu \sim (\Lambda, \Lambda, \Lambda)$	q_s, A_s^μ, h	S, $\overline{S}$
off-shell	$p_n^\mu + p_{\overline{n}}^\mu + p_s \sim (Q, Q, \Lambda)$	ψ_H, A_Q^μ	W_Q^X, $\overline{W}_Q^X$
	$p_n^\mu + p_s^\mu \sim (\Lambda, Q, \Lambda)$	ψ_n, ξ_X, A_{nX}^μ	W_X, S_X
	$p_{\overline{n}}^\mu + p_s^\mu \sim (Q, \Lambda, \Lambda)$	$\psi_{\overline{n}}$, χ_X, $A_{\overline{n}X}^\mu$	$\overline{W}_X$, $\overline{S}_X$

It appears in the Lagrangian for the collinear quarks. The explicit Lagrangian for the collinear quarks involving the off-shell fields of order $p^2 \sim Q\Lambda$ can be found in Eqs. (A4) and (A5) in Ref. [12].

The difference between the Lagrangians given in Eqs. (A.8) and (A.16) is that all the off-shell fields are labeled by X, and $A_n \to A_n^X + A_n$, $A_{\overline{n}} \to A_{\overline{n}}^X + A_{\overline{n}}$. Therefore the heavy quark field, after the off-shell fields are integrated out, is written as

$$h + \psi_n + \psi_{\overline{n}} + \psi_H = W_Q^X \overline{W}_X h, \tag{A.17}$$

and the collinear quarks are written as [12]

$$\xi + \xi_X + \xi_Q = \overline{W}_Q^X S_X \xi, \ \chi + \chi_X + \chi_Q = W_Q^X \overline{S}_X \chi, \tag{A.18}$$

where W_Q^X, and S_X are the Fourier transforms of the Wilson lines

$$\begin{aligned} W_Q^X(y) &= \mathrm{P}\exp\Big\{ig\int_{-\infty}^{y} ds\Big[\overline{n}\cdot A_Q(s\overline{n}) + \overline{n}\cdot A_{nX}(s\overline{n}) + \overline{n}\cdot A_n(s\overline{n})\Big]\Big\}, \\ S_X(y) &= \mathrm{P}\exp\Big\{ig\int_{-\infty}^{y} ds\Big[n\cdot A_{nX}(sn) + n\cdot A_s(sn)\Big]\Big\}, \end{aligned} \tag{A.19}$$

and $\overline{W}_Q^X$ and $\overline{S}_X$ are obtained by replacing n^μ by $\overline{n}^\mu$ in Eq. (A.19). The Wilson lines satisfy

$$\overline{W}_Q^{X\dagger} W_Q^X = W_X \overline{W}_X^\dagger, \ S_X^\dagger W_X = W S^\dagger, \ \overline{S}_X \overline{W}_X = \overline{W S}^\dagger. \tag{A.20}$$

The last two relations in Eq. (A.17) were obtained in Ref. [12] and the first relation is new.

By integrating out the off-shell fields, the singlet four-quark operator in $\mathrm{SCET_{II}}$ is given by

$$\begin{aligned} &\Big(\overline{\xi} + \overline{\xi}_X + \overline{\xi}_Q\Big)\Gamma_1\Big(h + \psi_n + \psi_{\overline{n}} + \psi_H\Big) \\ &\times\Big(\overline{\chi} + \overline{\chi}_X + \overline{\chi}_Q\Big)\Gamma_2\Big(\chi + \chi_X + \chi_Q\Big) \\ = &\Big(\overline{\xi} S_X^\dagger \overline{W}_Q^{X\dagger}\Gamma_1 W_Q^X \overline{W}_X h\Big)\cdot\Big(\overline{\chi}\overline{S}_X^\dagger W_Q^{X\dagger}\Gamma_2 W_Q^X \overline{S}_X \chi\Big) \\ = &\Big(\overline{\xi} S_X^\dagger W_X \Gamma_1 \overline{W}_X^\dagger \overline{W}_X h\Big)\cdot\Big(\chi\Gamma_2\chi\Big) = \overline{\xi} W \Gamma_1 S^\dagger h \cdot \overline{\chi}\Gamma_2\chi, \end{aligned} \tag{A.21}$$

using Eq. (A.20), and similarly the nonsinglet four-quark operator is given as

$$
\begin{aligned}
&\left(\overline{\xi}S_X^{\dagger}\overline{W}_Q^{X\dagger}\right)_{\beta}\Gamma_1\left(W_Q^X\overline{W}_X h\right)_{\alpha}\cdot\left(\overline{\chi}\overline{S}_X^{\dagger}W_Q^{X\dagger}\right)_{\alpha}\Gamma_2\left(W_Q^X\overline{S}_X\chi\right)_{\beta}\\
&=\left(\overline{\xi}S_X^{\dagger}W_X\right)_{\beta}\Gamma_1\left(\overline{S}_X^{\dagger}\overline{W}_X h\right)_{\alpha}\cdot\overline{\chi}_{\alpha}\Gamma_2\left(\overline{W}_X^{\dagger}\overline{S}_X\chi\right)_{\beta}\\
&=(\overline{\xi}WS^{\dagger})_{\beta}\Gamma_1(\overline{S}^{\dagger}h)_{\alpha}\cdot(\overline{\chi}\overline{W})_{\alpha}\Gamma_2(\overline{S}\overline{W}^{\dagger}\chi)_{\beta}\\
&=\left(\overline{\xi}WS^{\dagger}\overline{S}\right)_{\beta}\Gamma_1(\overline{S}^{\dagger}h)_{\alpha}\cdot(\overline{\chi}\overline{W})_{\alpha}\Gamma_2(\overline{W}^{\dagger}\chi)_{\beta}. \qquad \text{(A.22)}
\end{aligned}
$$

This result is the same as the explicit calculation obtained in Section 2, and it is a proof to all orders in α_s using the auxiliary field method.

B Derivation of the subleading operators using the auxiliary field method

We can also derive the subleading operators $O_i^{(1a,1b)}$ in Eq. (36) and $J_i^{(1a)}$ and $J_i^{(1b)}$ in Eq. (53) using the auxiliary field method. Here we have to consider off-shell modes from the collinear gluons both in the n^μ and $\overline{n}^\mu$ directions. For simplicity, we consider the subleading operators in $\mathrm{SCET_I}$ disregarding the off-shell modes by soft gluons. The leading-order result was derived in Ref. [12], and here we present a new result which yields gauge-invariant operators at subleading order. In addition to the solutions in Eq. (A.3) at leading order, when we include the subleading terms in the covariant derivative in Eq. (A.2), we obtain the equations of motion at subleading order, which are given as

$$
\begin{aligned}
&[W_Q\overline{n}\cdot\mathcal{P}W_Q^{\dagger},[\overline{W}_Q n\cdot\mathcal{Q}\overline{W}_Q^{\dagger},i\mathcal{D}_{Q\perp}^{\nu}]]=0,\\
&[\overline{W}_Q n\cdot\mathcal{Q}\overline{W}_Q^{\dagger},[W_Q\overline{n}\cdot\mathcal{P}W_Q^{\dagger},i\mathcal{D}_{Q\perp}^{\nu}]]=0, \qquad \text{(B.1)}
\end{aligned}
$$

where $i\mathcal{D}_{Q\perp}^{\mu}=iD_{Q\perp}^{\mu}+gA_{Q\perp}^{\mu}$. Here we make an ansatz

$$
\overline{W}_Q^{\dagger}i\mathcal{D}_{n\perp Q}^{\nu}=iD_{n\perp}^{\nu}\overline{W}_Q^{\dagger}\ \text{or}\ W_Q^{\dagger}i\mathcal{D}_{\overline{n}\perp Q}^{\nu}=iD_{\overline{n}\perp}^{\nu}W_Q^{\dagger}, \qquad \text{(B.2)}
$$

and they satisfy Eq. (B.1).

First the intermediate form of the subleading four-quark operators relevant to the nonfactorizable spectator contribution and the heavy-to-light form factor can be obtained by neglecting the off-shell modes in the $\overline{n}^\mu$ direction since we are interested in the operators proportional to gA_n^μ at leading order in g. As a definite example, let us concentrate on the operator $O_i^{(1b)}$ and obtain the gauge-invariant form by integrating out the off-shell modes. The derivation for the operator $O_i^{(1a)}$ can be done in a similar way. In $\mathrm{SCET_I}$, without the off-shell modes in the $\overline{n}^\mu$ direction, the operator $O_i^{(1b)}$ is given by

$$
\begin{aligned}
O_i^{(1b)} =& (\overline{\xi} + \overline{\xi}_Q)\Gamma_{1i}(h + \psi_n + \psi_{\overline{n}} + \psi_H) \\
&\times(\overline{\chi} + \overline{\chi}_Q)\Gamma_{2i}\Big(\overline{W}_Q \frac{1}{n\cdot\mathcal{Q}} \overline{W}_Q^\dagger\Big)[i\not{D}_{n\perp} + g\not{A}_{Q\perp}]\frac{\not{\overline{n}}}{2}(\chi + \chi_Q),
\end{aligned} \tag{B.3}
$$

where the color indices are suppressed. When we include off-shell modes, each field is written as

$$
h + \psi_n + \psi_{\overline{n}} = W_Q \overline{W} h,\ \ \xi + \xi_Q = \overline{W}_Q \xi,\ \ \chi + \chi_Q = W_Q \chi. \tag{B.4}
$$

Then the operator in Eq. (B.3) is written as

$$
\Big[(\overline{\xi}\overline{W}_Q^\dagger)\Gamma_{1i}(W_Q\overline{W}h)\Big]\cdot\Big[(\overline{\chi}W_Q^\dagger)\Gamma_{2i}\overline{W}_Q\frac{1}{n\cdot\mathcal{Q}}\overline{W}_Q^\dagger\Big([i\not{\mathcal{D}}_{n\perp Q}W_Q]\frac{\not{\overline{n}}}{2}\chi\Big)\Big]. \tag{B.5}
$$

If we use the ansatz in Eqs. (A.4) and (B.2), the singlet operator becomes

$$
\begin{aligned}
O_S^{(1b)} =& \Big[(\overline{\xi}\overline{W}_Q^\dagger W_Q)_\alpha \Gamma_1 (\overline{W}h)_\alpha\Big] \\
&\times\Big[(\overline{\chi}W_Q^\dagger)_\beta\Gamma_2\Big(\overline{W}_Q\frac{1}{n\cdot\mathcal{Q}}\overline{W}_Q^\dagger[i\not{\mathcal{D}}_{n\perp Q}W_Q]\frac{\not{\overline{n}}}{2}\chi\Big)_\beta\Big] \\
=& \Big[(\overline{\xi}W)_\alpha\Gamma_1 h_\alpha\Big] \\
&\times\Big[(\overline{\chi}W_Q^\dagger\overline{W}_Q)_\beta\Gamma_2\frac{1}{n\cdot\mathcal{Q}}\Big([i\not{\mathcal{D}}_{n\perp}\overline{W}_Q^\dagger W_Q]\frac{\not{\overline{n}}}{2}\chi\Big)_\beta\Big] \\
=& \Big[(\overline{\xi}W)_\alpha\Gamma_1 h_\alpha\Big]\cdot\Big[(\overline{\chi}\overline{W})_\beta\Gamma_2\frac{1}{n\cdot\mathcal{Q}}\Big([W^\dagger i\not{\mathcal{D}}_{n\perp}W]\frac{\not{\overline{n}}}{2}\overline{W}^\dagger\chi\Big)_\beta\Big],
\end{aligned} \tag{B.6}
$$

and the nonsinglet operator becomes

$$
\begin{aligned}
O_N^{(1b)} =& \Big[(\overline{\xi}\overline{W}_Q^\dagger)_\beta\Gamma_1(W_Q\overline{W}h)_\alpha\Big]\cdot\Big[(\overline{\chi}W_Q^\dagger)_\alpha\Gamma_2\Big(\overline{W}_Q\frac{1}{n\cdot\mathcal{Q}}\overline{W}_Q^\dagger[i\not{\mathcal{D}}_{n\perp Q}W_Q]\frac{\not{\overline{n}}}{2}\chi\Big)_\beta\Big] \\
=& \Big[\overline{\xi}_\beta\Gamma_1(\overline{W}h)_\alpha\Big]\cdot\Big[\overline{\chi}_\alpha\Gamma_2\frac{1}{n\cdot\mathcal{Q}}\Big(\overline{W}_Q^\dagger[i\not{\mathcal{D}}_{n\perp Q}W_Q]\frac{\not{\overline{n}}}{2}\chi\Big)_\beta\Big] \\
=& \Big[\overline{\xi}_\beta\Gamma_1(\overline{W}h)_\alpha\Big]\cdot\Big[\overline{\chi}_\alpha\Gamma_2\frac{1}{n\cdot\mathcal{Q}}\Big([i\not{\mathcal{D}}_{n\perp}\overline{W}_Q^\dagger W_Q]\frac{\not{\overline{n}}}{2}\chi\Big)_\beta\Big] \\
=& \Big[\Big(\overline{\xi}W\Big)_\beta\Gamma_1 h_\alpha\Big]\cdot\Big[\Big(\overline{\chi}\overline{W}\Big)_\alpha\Gamma_2\Big([W^\dagger i\not{\mathcal{D}}_{n\perp}W]\frac{1}{n\cdot\mathcal{Q}}\frac{\not{\overline{n}}}{2}\overline{W}^\dagger\chi\Big)_\beta\Big],
\end{aligned} \tag{B.7}
$$

These are the subleading operators which are gauge invariant. Other operators can be shown to have the forms presented in the paper in a similar way.

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